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Institute of Media, Information, and Network

Continuous-Time Fourier Transform

Hongkai Xiong
熊红凯
http://min.sjtu.edu.cn

Department of Electronic Engineering
Shanghai Jiao Tong University

LTI Systems

$$x_k[n] \to y_k[n] \;\Longrightarrow\; \sum_k a_k x_k[n] \to \sum_k a_k y_k[n]$$

- **If we can find a set of "basic" signals, such that**
  - a rich class of signals can be **represented as linear combinations** of these basic (building block) signals.
  - the **response** of LTI Systems to these basic signals are both **simple and insightful**

- **Candidate sets of "basic" signals**
  - Unit impulse function and its delays: $\delta(t)/\delta[n]$
  - Complex exponential/sinusoid signals: $e^{st}, e^{j\omega t}/z^n, e^{j\omega n}$

Fourier Series of CT Periodic Signals

- Recall a **periodic** CT signal $x(t)$
  - with fundamental period T, and
  - fundamental frequency $\omega_0 = 2\pi/T$

- **Fourier series** represent $x(t)$ in terms of harmonically related complex exponentials
  - **Synthesis equation**

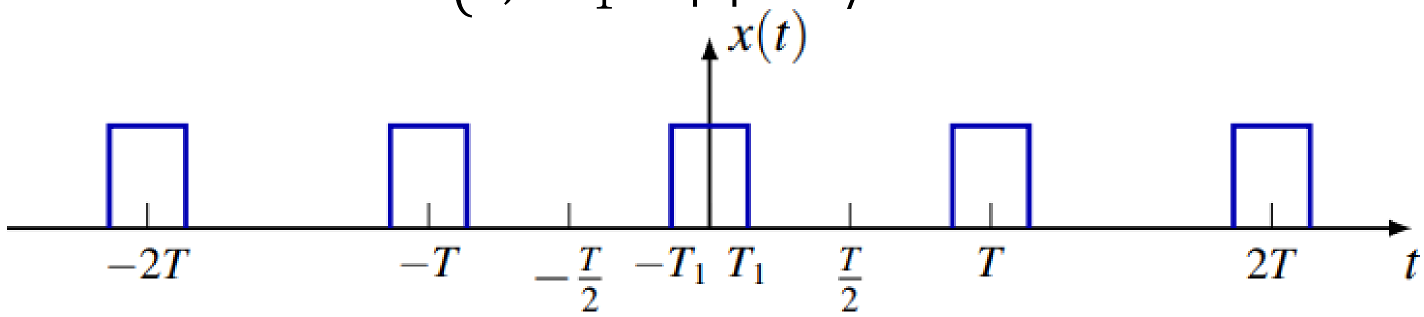
$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}}$$

  - **Analysis equation**

$$\boxed{a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\,dt}$$

Motivating Example: Periodic Square Wave

- In one period, $x_T(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$



- **Fourier coefficients**

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} 1\,dt = \frac{2T_1}{T}, \qquad a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}, k \neq 0$$

- Frequency component at $\omega_k = k\omega_0$ satisfies

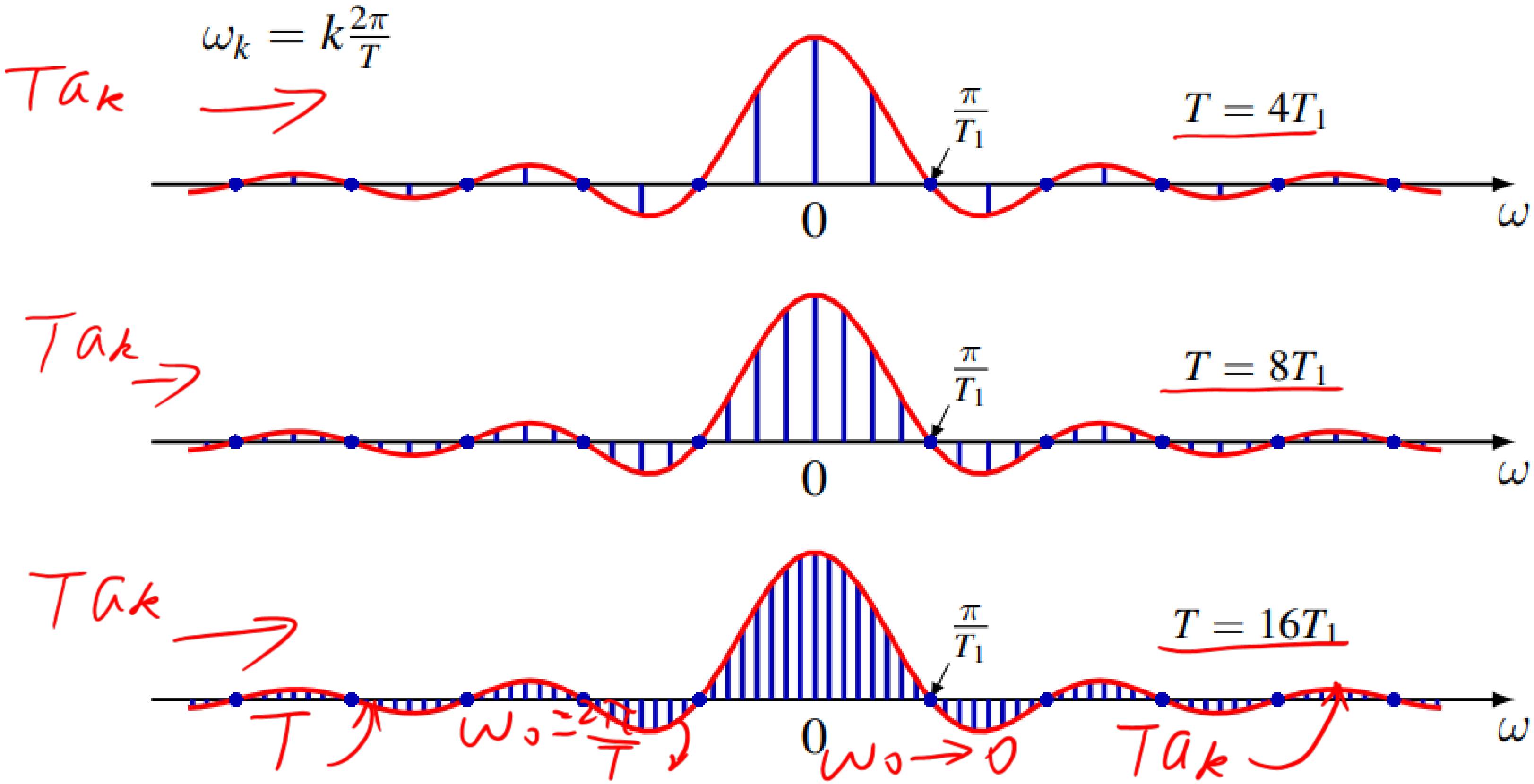
$$Ta_k = \left.\frac{2\sin(\omega T_1)}{\omega}\right|_{\omega = k\omega_0}$$

$X(j\omega) = \frac{2\sin\omega T_1}{\omega}$

Ta_k thought as **sampled** value of **envelope** $X(j\omega) \triangleq 2\sin(\omega T_1)/\omega$ at $\omega_k = k\omega_0$

Motivating Example: Periodic Square Wave

- $X(j\omega_k) \triangleq 2\sin(\omega_k T_1)/\omega_k$ for fixed T_1 and increasing T



- **Observation:** as $T \to \infty$, discrete frequencies sampled more densely

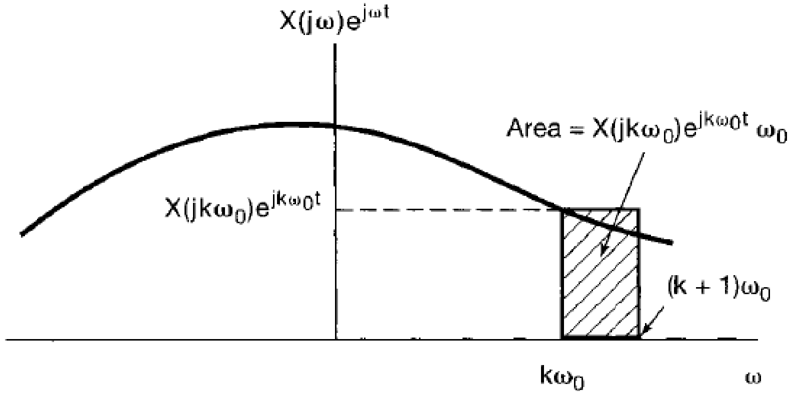
Motivating Example: Periodic Square Wave

- **As $\boldsymbol{T \to \infty}$, $\boldsymbol{x_T(t) \to x(t) \triangleq u\left(t+\frac{T_1}{2}\right) - u(t-\frac{T_1}{2})}$**, a rectangular impulse

$$x_T(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0$$

$$\Longrightarrow \lim_{T\to\infty} x_T(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$



- Thus

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Motivating Example: Periodic Square Wave

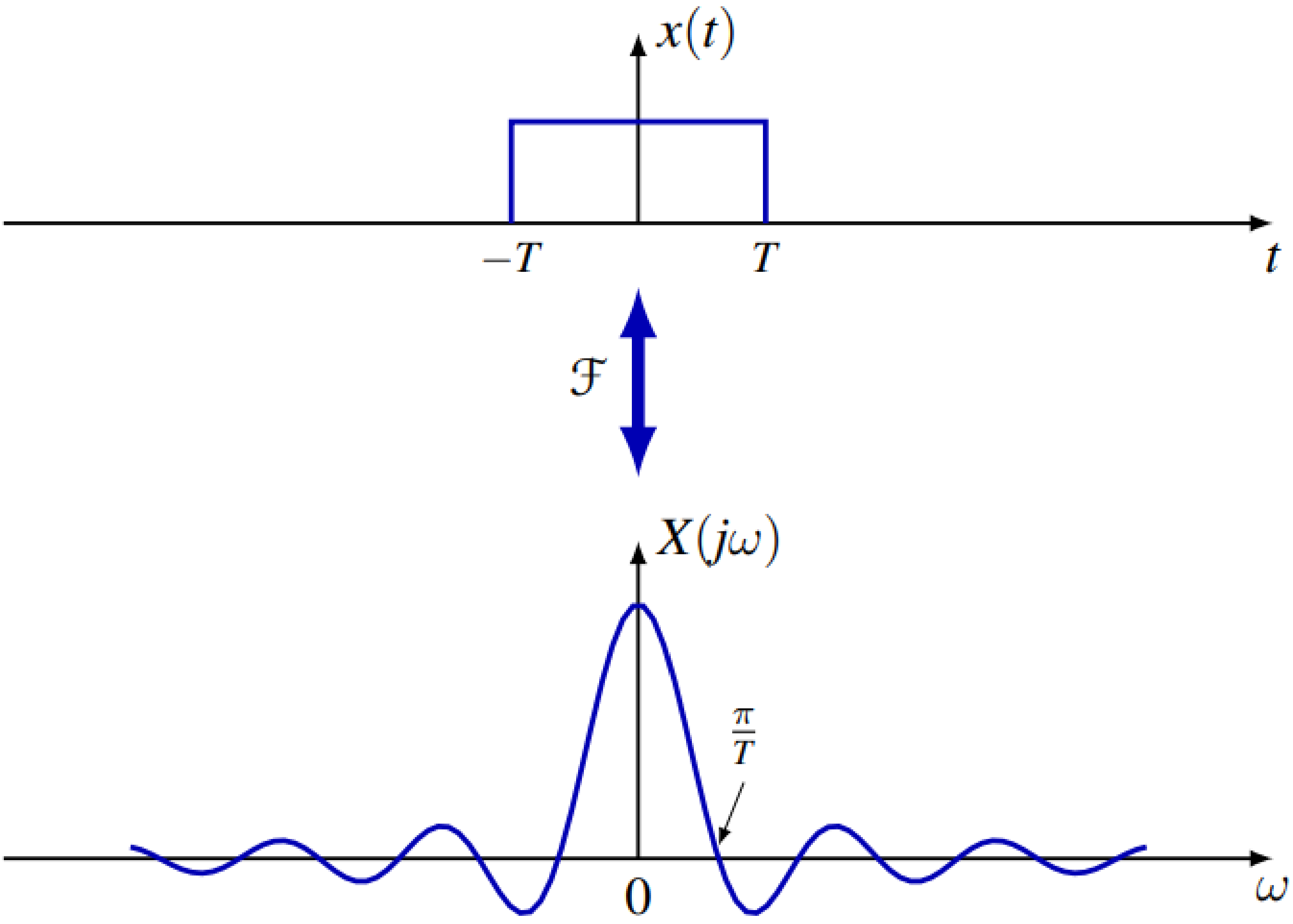
- For **envelope $X(j\omega)$**

$$
\begin{aligned}
X(j\omega_k) &= Ta_k \\
&= \int_{-\frac{T}{2}}^{\frac{T}{2}} x_T(t)e^{-jk\omega t}dt \\
&= \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-j\omega_k t}dt \\
&= \int_{-\infty}^{\infty} x(t)e^{-j\omega_k t}dt
\end{aligned}
$$

- Thus

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Motivating Example: Periodic Square Wave



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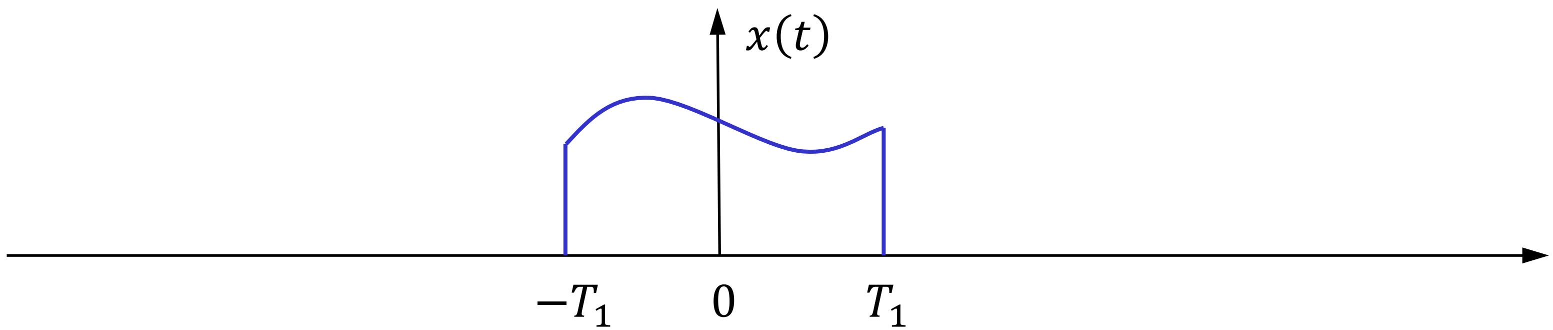
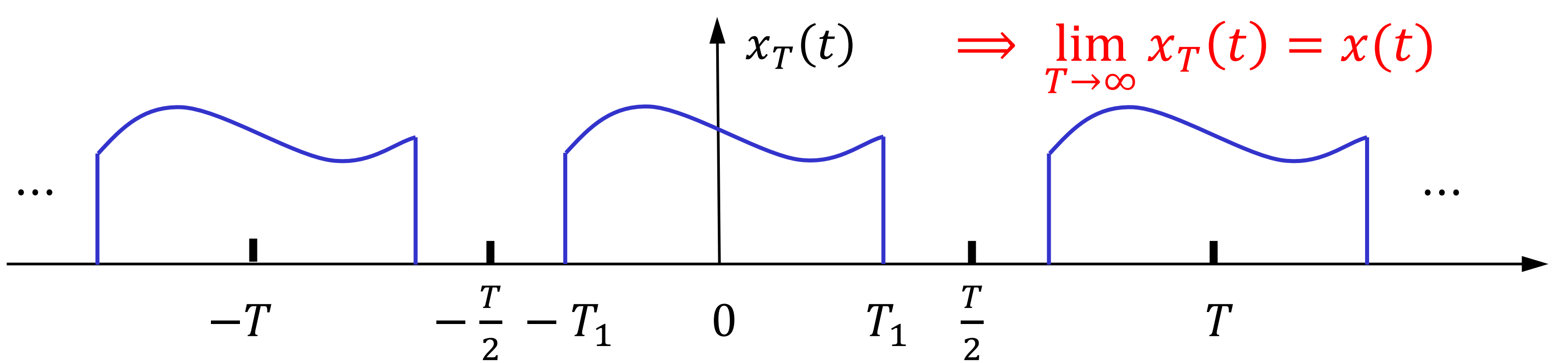
CT Fourier Transform of Aperiodic Signal $x(t)$

- **General strategy**
  - approximate $x(t)$ by a periodic signal $x_T(t)$ with infinite period T





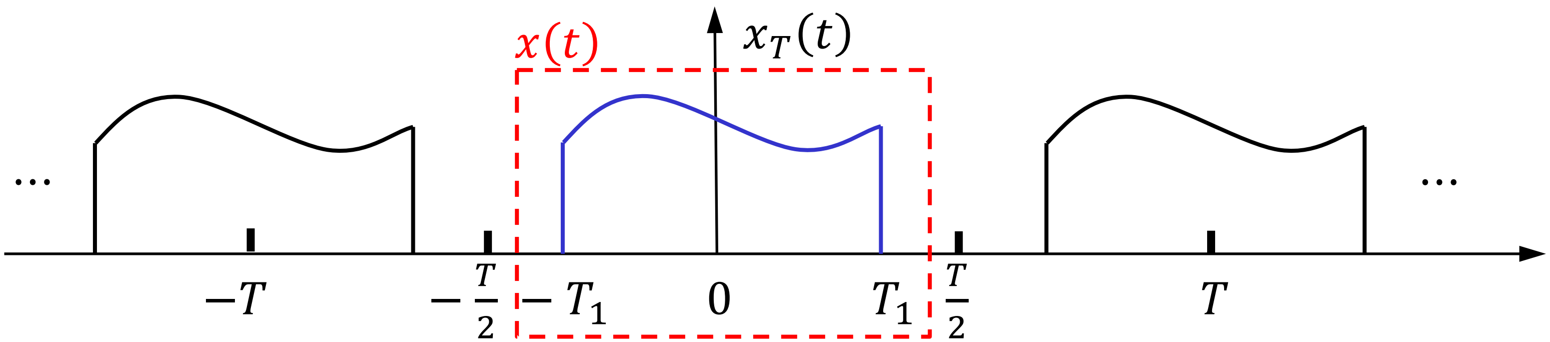
CT Fourier Transform of Aperiodic Signal $x(t)$

- **Step 1:** for aperiodic signal $x(t)$ with sup $x(t) \subset [-T_1, T_1]$ **construct a periodic signal** $x_T(t)$, for which one period is $x(t)$

$$x_T(t) = \sum_{k=-\infty}^{\infty} x(t - kT)$$

  - Then

$$x(t) = x_T(t), \forall |t| < T/2$$



CT Fourier Transform of Aperiodic Signal $x(t)$

- **Step 2:** determine the **Fourier series representation** for $x_T(t)$
  - **Fourier series coefficients**

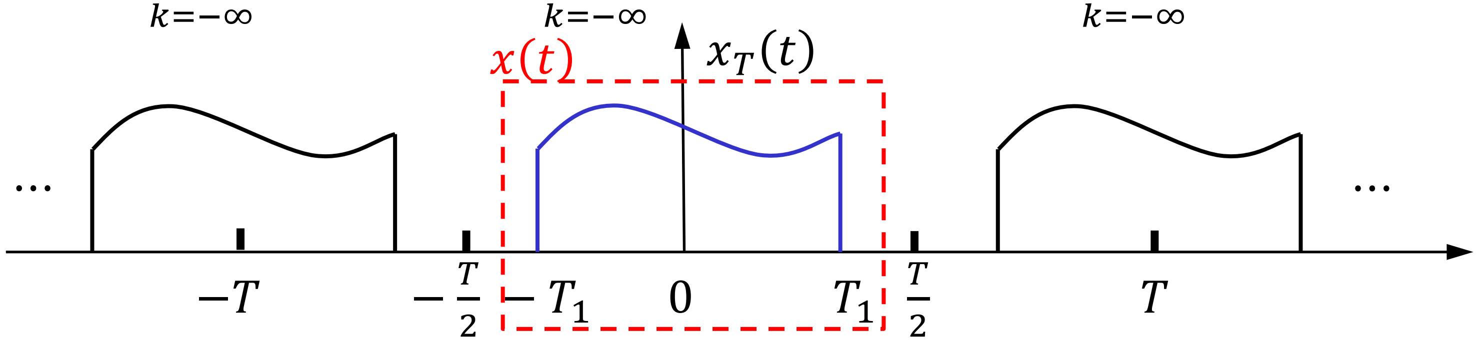
$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt$$

  - Define the **envelope** of Ta_k

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

  - Then

$$x_T(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0$$



CT Fourier Transform of Aperiodic Signal $x(t)$

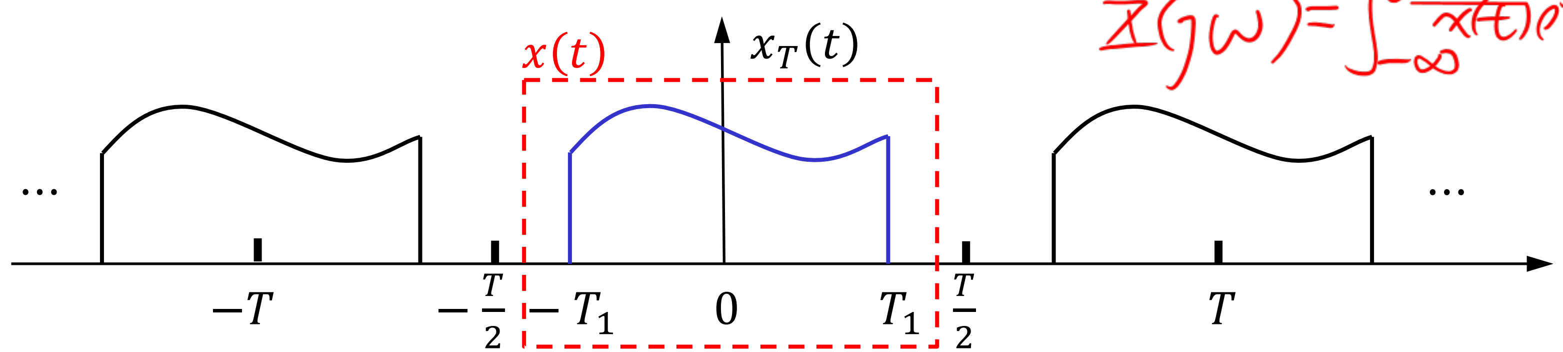
- **Step 3:** approximate $x(t)$ with $x_T(t)$ by increasing $T \to \infty$
  - As $T \to \infty \implies \omega_0 \to 0, x_T(t) \to x(t), \omega_0 \to d\omega, \sum \to \int$

$$x_T(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0$$

$$\Downarrow$$

$$x(t) = \lim_{T\to\infty} x_T(t) = \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$



CT Fourier Transform Pair

- **Fourier transform (analysis equation)**

$$X(j\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

where $X(j\omega)$ called **spectrum** of $x(t)$

- **Inverse Fourier transform (synthesis equation)**

$$x(t) = \mathcal{F}^{-1}\{X(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Superposition of complex exponentials at **continuum** of frequencies, frequency component $e^{j\omega t}$ has “amplitude” of $X(j\omega)d\omega/2\pi$

CT Fourier Series of Periodic Signal $x_T(t)$

- **Fourier Transform** of a finite duration signal $x(t)$ that is equal to $x_T(t)$ over one period, e.g., for $t_0 \le t \le t_0 + T$

$$X(j\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

- **Fourier coefficients** of periodic signal $x_T(t)$ with period T

$$a_k = \frac{1}{T}\int_{t_0}^{t_0+T} x_T(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt$$
$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} X(j\omega)\Big|_{\omega = k\omega_0}$$

$\Longrightarrow$ **proportional** to **samples** of the **Fourier transform** of $x(t)$, i.e., one period of $x_T(t)$

Convergence of Fourier Transform

- Suppose from $x(t)$, we have

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- Let $\hat{x}(t)$ be the **Fourier transform representation** of $x(t)$, obtained using the synthesis equation

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

- **Question:** when can $\hat{x}(t)$ be a valid representation of the original signal $x(t)$?

Finite Energy Condition

- One useful notion for engineers:
  - **no energy** in **approximation error**

$$e(t) \triangleq x(t) - \hat{x}(t) = x(t) - \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

and $$\int_{-\infty}^{\infty} |e(t)|^2 dt = 0$$

$x(t) = \hat{x}(t), \forall t$

$e(t) = x(t) - \hat{x}(t) = 0, \forall t$

- **Finite energy condition (square integrable)**

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

$\Rightarrow X(j\omega) < \infty$

  - **Note:** does not guarantee $x(t)$ equal to its Fourier representation at any time t, but guarantees no energy in approximation error

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Dirichlet Conditions

$x(t) = \hat{x}(t)$, except discontinuities

- **Condition 1.**
  - $x(t)$ is **absolutely integrable**, i.e.,

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

$x_T(t)$

- **Condition 2.**
  - Within any finite interval, $x(t)$ has a finite number of maxima and minima

- **Condition 3.**
  - Within any finite interval, $x(t)$ has only a finite number of discontinuities with finite values

- $\Longrightarrow$ **absolutely integrable** signals that are **continuous** or with **a finite number of discontinuities** have Fourier transforms

Example: Right-sided Decaying Exponential

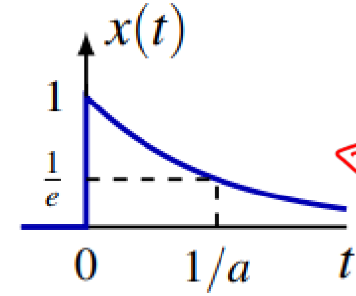
- $x(t) = e^{-at}u(t), a > 0$ is a real number

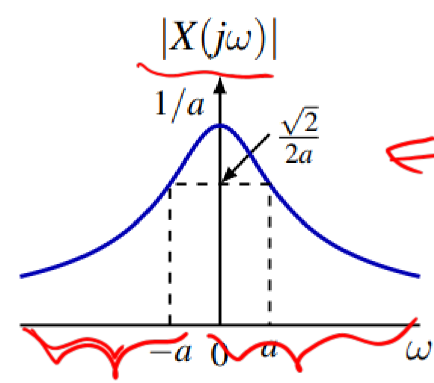
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt$$

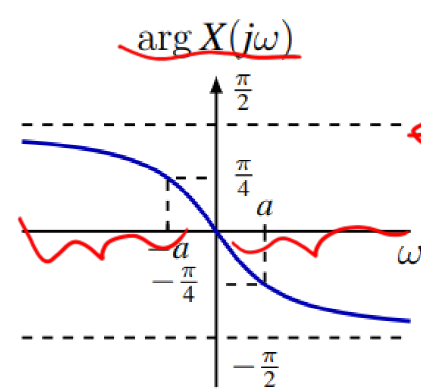
$$= -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{1}{a+j\omega}$$

$a>0$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}, \arg X(j\omega) = -\arctan\frac{\omega}{a}$$







- **Observation:**
  - **Magnitude** spectrum: **even** symmetry
  - **Phase** spectrum: **odd** symmetry

$X(j\omega) = \frac{1}{a+j\omega}$

- **Note:** also works for complex a with $\mathcal{Re}\{a\} > 0$

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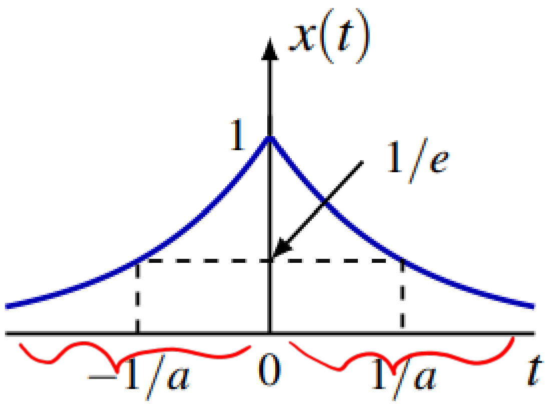
Example: Two-sided Decaying Exponential

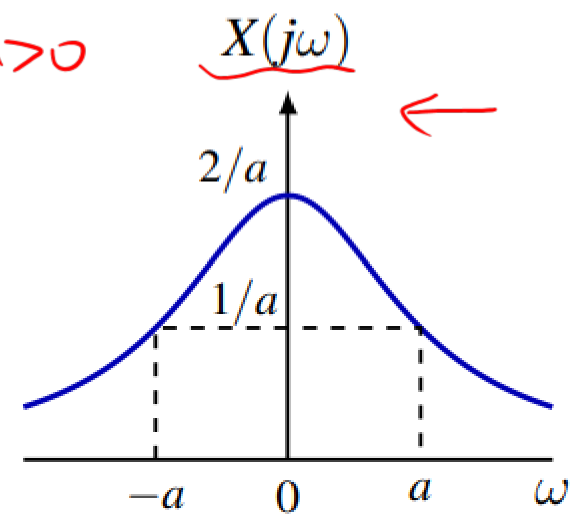
- $x(t) = e^{-|a|t}u(t)$, $a > 0$ is a real number

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$
$$= \int_{-\infty}^{\infty} e^{-|a|t}e^{-j\omega t}dt$$
$$= \int_{-\infty}^{0} e^{at}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt$$
$$= \frac{1}{a-j\omega}e^{(a-j\omega)t}\Big|_{-\infty}^{0} - \frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_{0}^{\infty}$$
$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$

real

$a>0$





- **Note:** also works for complex a with $\mathcal{Re}\{a\} > 0$

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Example: Rectangular Pulse

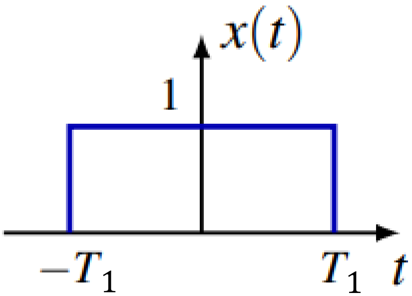
- $x(t) = u(t+T_1) - u(t-T_1)$
- **Fourier transform**

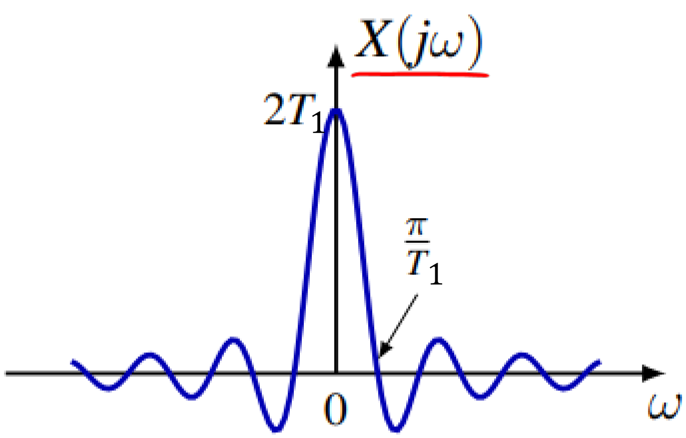
$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}dt = 2\frac{\sin(\omega T_1)}{\omega}$$

- **Inverse Fourier transform**

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\frac{\sin(\omega T_1)}{\omega}e^{j\omega t}d\omega$$

- As $T_1 \to \infty$, (define $\mathrm{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$)
  - Frequency domain: $2\pi\frac{\sin(\omega T_1)}{\pi\omega} = 2\pi\frac{T_1}{\pi}\mathrm{sinc}\left(\frac{\omega T_1}{\pi}\right) \to 2\pi\delta(\omega)$
  - Time domain: $x(t) \to 1$, DC





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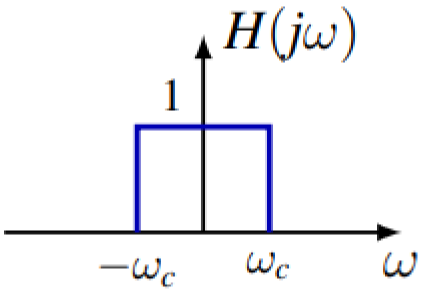
Example: Ideal Lowpass Filter

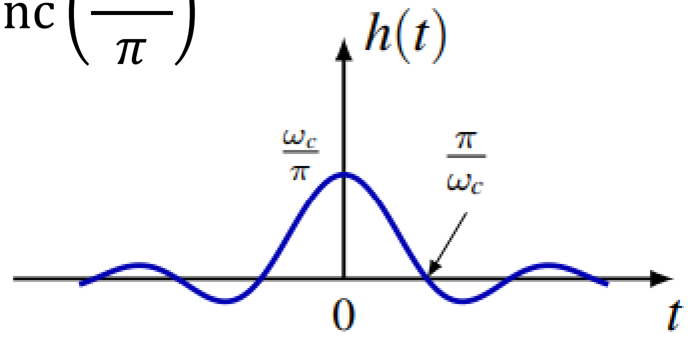
- **Frequency response**

$$H(j\omega) = u(\omega + \omega_c) - u(\omega - \omega_c)$$



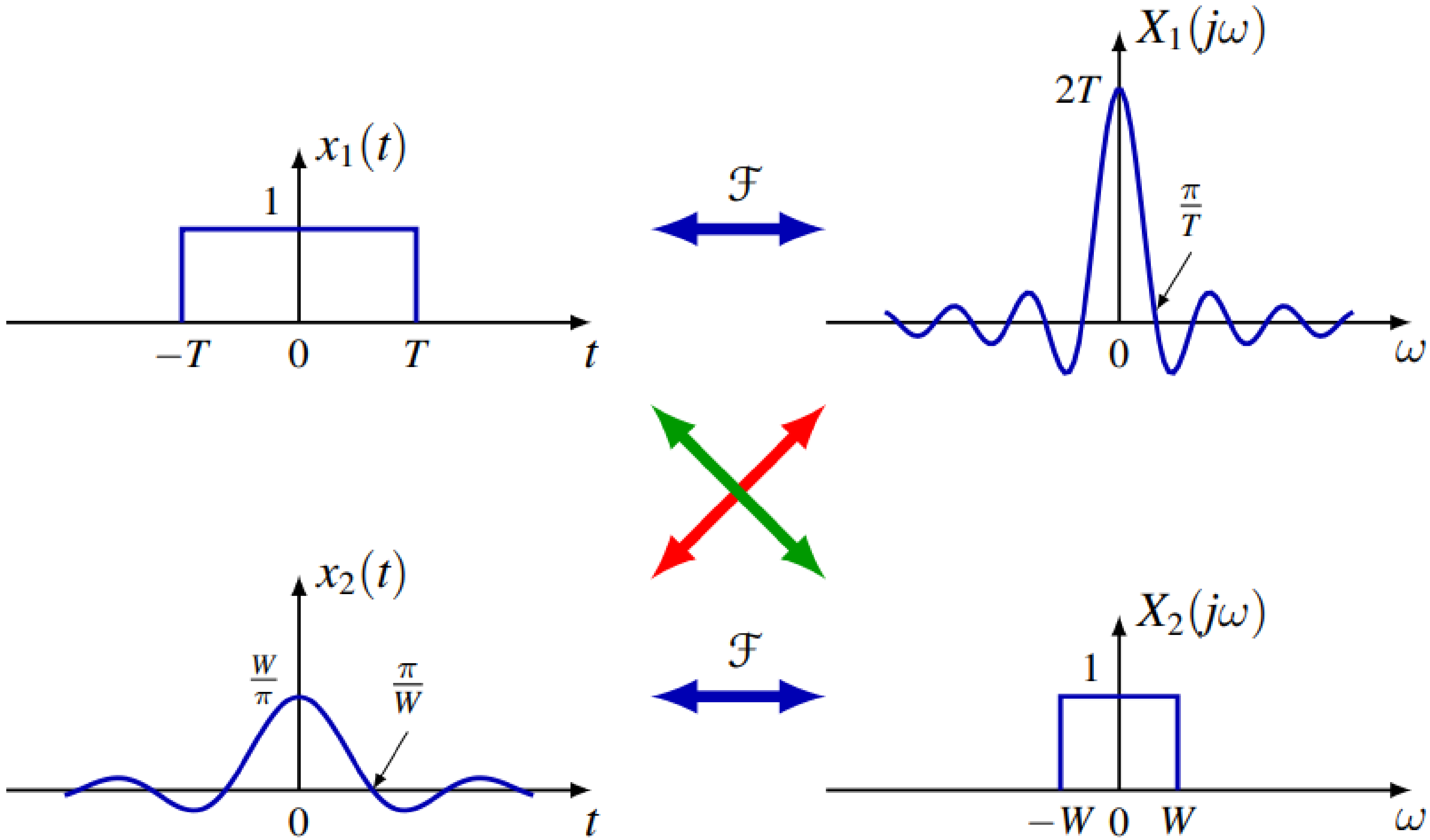
- **Impulse response**

$$h(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}d\omega = \frac{\sin(\omega_c t)}{\pi t} = \frac{\omega_c}{\pi}\text{sinc}\left(\frac{\omega_c t}{\pi}\right)$$



- As $\omega_c \to \infty$,
  - Time domain
    - $h(t) \to \delta(t)$, becomes identity system
  - Frequency domain
    - $H(j\omega) \to 1$, passes all frequencies

Duality



$x_1(t)$
1
$-T$ 0 T t
$\mathcal{F}$
$X_1(j\omega)$
$2T$
$\frac{\pi}{T}$
0 ω
$x_2(t)$
$\frac{W}{\pi}$
$\frac{\pi}{W}$
0 t
$\mathcal{F}$
$X_2(j\omega)$
1
$-W$ 0 W ω

Examples

- **Unit impulse**

$$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = 1$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} d\omega = \lim_{\omega_c \to \infty} \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t} d\omega = \lim_{\omega_c \to \infty} \frac{\sin(\omega_c t)}{\pi t} = \delta(t)$$

$\delta(t)$ has "white" spectrum, equal amount of all frequencies!

- **DC Signal**

$$x(t) = 1$$

$$X(j\omega) = \lim_{T_1 \to \infty} \int_{-T_1}^{T_1} e^{-j\omega t} dt = \lim_{T_1 \to \infty} \left[2\pi \frac{\sin(\omega T_1)}{\pi\omega} \right] = 2\pi\delta(\omega)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega) e^{j\omega t} d\omega = 1$$

Spectrum of DC signal is impulse at zero frequency!

Fourier Transform of Periodic Signals

- A periodic signal $x(t)$ with fundamental frequency $\omega_0 = 2\pi/T$ has a **Fourier series** representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\Longrightarrow \; X(j\omega) = \mathcal{F}\{x(t)\} = \sum_{k=-\infty}^{\infty} a_k \mathcal{F}\{e^{jk\omega_0 t}\}$$

- Question then becomes

$$\mathcal{F}\{e^{jk\omega_0 t}\} = ?$$

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Fourier Transform of Periodic Signals

- Using the **basic Fourier transform pair**

$$e^{j\omega_0 t} \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega-\omega_0)$$

  - $\mathcal{F}\{e^{j\omega_0 t}\} = \lim_{T_1\to\infty}\int_{-T_1}^{T_1} e^{-j(\omega-\omega_0)t}dt = \lim_{T_1\to\infty}\left\{2\pi\frac{\sin[(\omega-\omega_0)T_1]}{\pi(\omega-\omega_0)}\right\}$
    $= 2\pi\delta(\omega-\omega_0)$
  - $\mathcal{F}^{-1}\{2\pi\delta(\omega-\omega_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$

- Thus, the **Fourier transform** of periodic signal $x(t)$ is

$$X(j\omega) = \mathcal{F}\{x(t)\} = \sum_{k=-\infty}^{\infty} a_k\mathcal{F}\{e^{jk\omega_0 t}\}$$

$$= \sum_{k=-\infty}^{\infty} 2\pi a_k\delta(\omega-k\omega_0)$$

Fourier Transform of Periodic Signals

- A periodic signal $x(t)$ with fundamental frequency $\omega_0 = 2\pi/T$
- **Fourier series**

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt$$

- **Fourier transform**

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

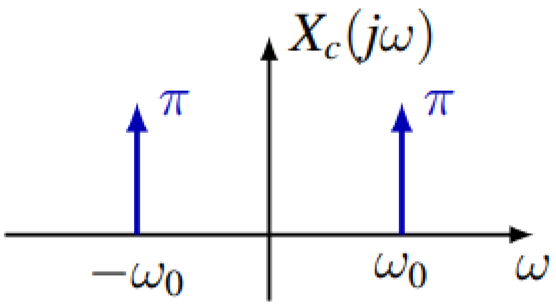
  - **Spectrum of periodic signal consists of a impulse train occurring at harmonically related frequencies!**
  - **Areas of impulses are 2π times Fourier series coefficients**

Example: Cosine and Sine

- $x_c(t) = \cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$

  $\Longrightarrow a_1 = 1, a_{-1} = 1,\ a_k = 0，\ k \neq \pm 1$

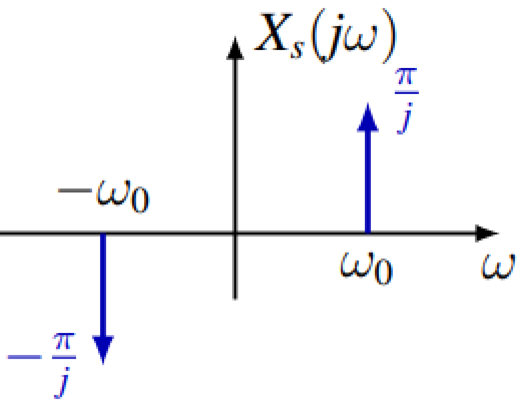
  $\Longrightarrow X_c(j\omega) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$



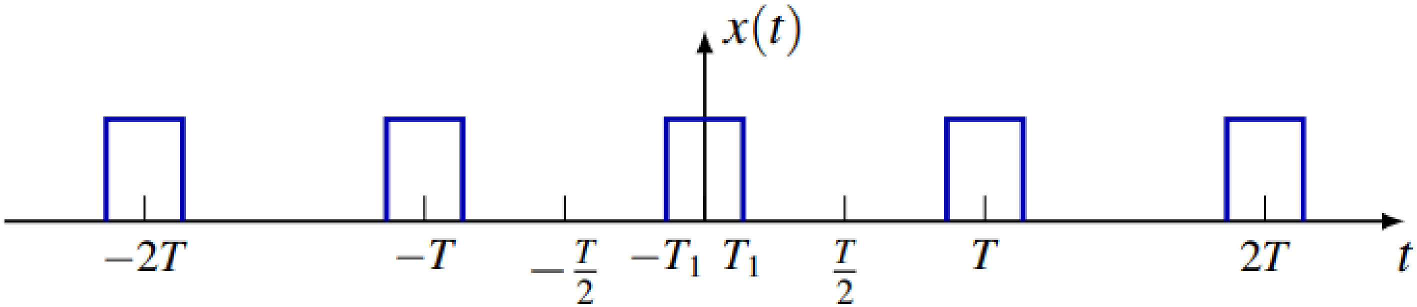
- $x_s(t) = \sin(\omega_0 t) = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$

  $\Longrightarrow a_1 = \frac{1}{2j}, a_{-1} = -\frac{1}{2j}, a_k = 0，\ k \neq \pm 1$

  $\Longrightarrow X_s(j\omega) = \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$



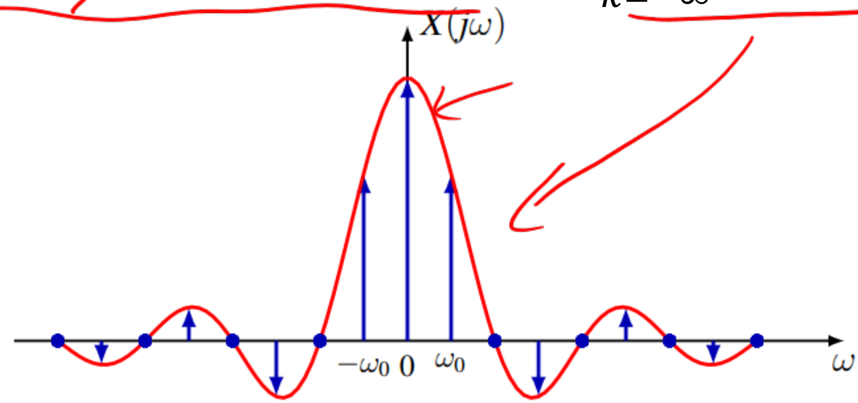
Example: Periodic Square Wave



$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} 1\,dt = \frac{2T_1}{T}, \qquad a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}, k \neq 0$$

ω_0

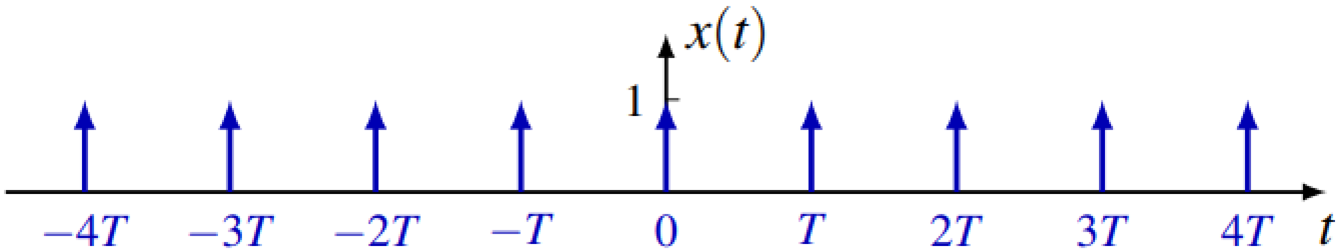
$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi \frac{2\sin(k\omega_0 T_1)}{k\cancel{\omega_0}T}\delta(\omega - k\omega_0) = \sum_{k=-\infty}^{\infty} \frac{2\sin(k\omega_0 T_1)}{k}\delta(\omega - k\omega_0)$$

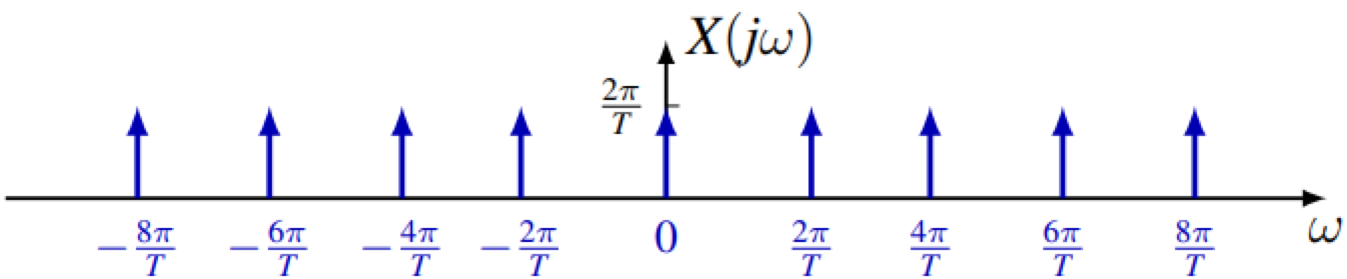


Example: Periodic Impulse Train

- $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$

$$\Rightarrow a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-jk\omega_0 t} dt = \frac{1}{T} \Rightarrow X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - k\frac{2\pi}{T})$$





Properties of CT Fourier Transform

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \quad y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$$

- **Linearity**

$$ax(t) + by(t) \overset{\mathcal{F}}{\longleftrightarrow} aX(j\omega) + bY(j\omega)$$

- **Time shifting**

$$x(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$

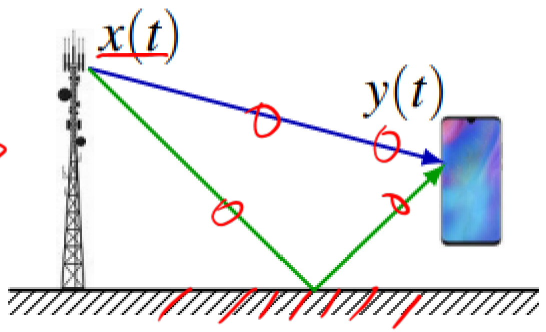
- **Frequency shifting**

$$e^{j\omega_0 t} x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j(\omega - \omega_0))$$

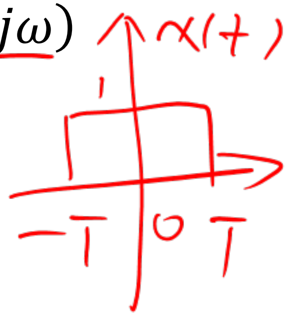
Example: Multipath Fading Effect

Transmitter: $x(t)$

Receiver: $y(t) = a_0 x(t) + a_1 x(t-\tau)$

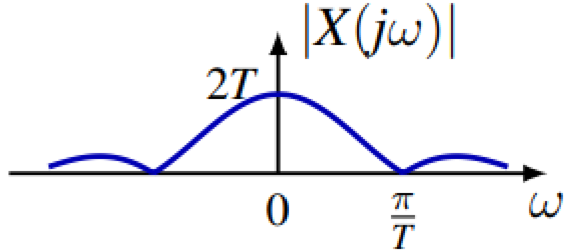


- $Y(j\omega) = a_0 X(j\omega) + e^{-j\omega\tau} a_1 X(j\omega)$
  $= (a_0 + e^{-j\omega\tau} a_1) X(j\omega)$



- Let $x(t) = u(t+T) - u(t-T)$

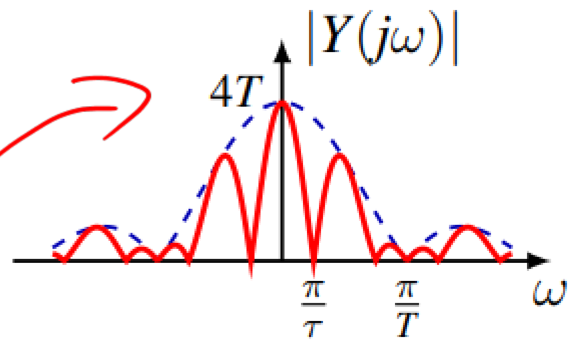
$$X(j\omega) = \frac{2\sin(\omega T)}{\omega}$$



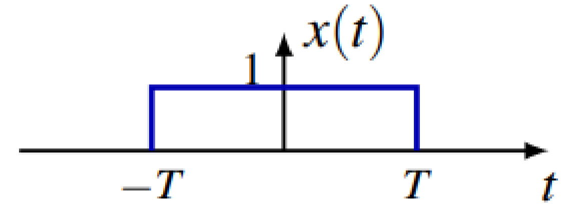
$$Y(j\omega) = (a_0 + e^{-j\omega\tau} a_1)\frac{2\sin(\omega T)}{\omega}$$

- When $a_0 = a_1 = 1$

$$Y(j\omega) = 4e^{-j\omega\tau/2}\cos(\omega\tau/2)\frac{2\sin(\omega T)}{\omega}$$

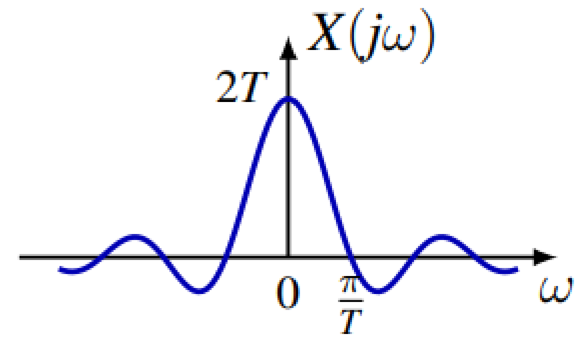


Example: Modulation



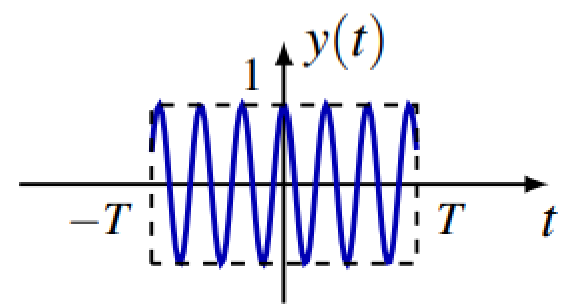
Baseband signal: $x(t)$

Modulated signal: $y(t) = x(t)\cos(\omega_c t)$



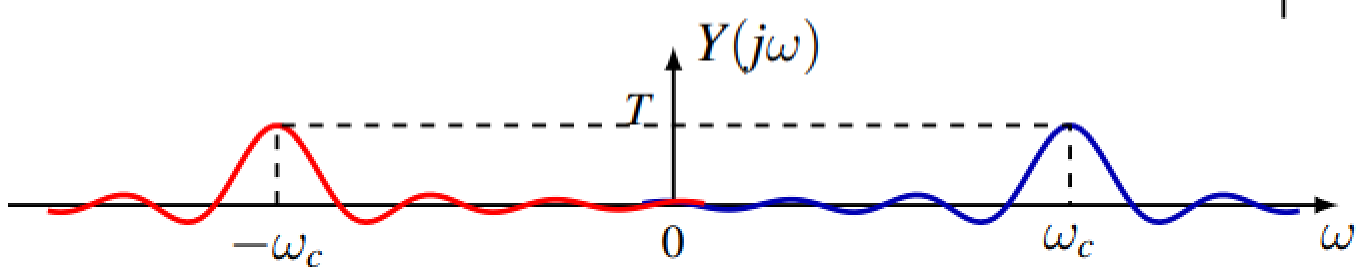
- $y(t) = \frac{1}{2}x(t)(e^{j\omega_c t} + e^{-j\omega_c t})$

  $$Y(j\omega) = \frac{1}{2}X(j(\omega - \omega_c)) + \frac{1}{2}X(j(\omega + \omega_c))$$



- Let $x(t) = u(t+T) - u(t-T)$

  $$Y(j\omega) = \frac{\sin((\omega - \omega_c)T)}{\omega - \omega_c} + \frac{\sin((\omega + \omega_c)T)}{\omega + \omega_c}$$



Properties of CT Fourier Transform

- **Time reversal**

$$\boxed{x(-t) \overset{\mathcal{F}}{\longleftrightarrow} X(-j\omega)}$$

- **Conjugation**

$$\boxed{x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-j\omega)}$$

Proof: $\mathcal{F}\{x^*(t)\} = \int_{-\infty}^{+\infty} x^*(t)e^{-j\omega t}dt = \left[\int_{-\infty}^{+\infty} x(t)e^{j\omega t}dt\right]^* = X^*\,(-j\omega)$

- **Conjugate Symmetry**
  - $x(t)$ real $\Leftrightarrow X(-j\omega) = X^*(j\omega)$
  - $x(t)$ even $\Leftrightarrow X(j\omega)$ even, $x(t)$ odd $\Leftrightarrow X(j\omega)$ odd
  - $x(t)$ real and even $\Leftrightarrow X(j\omega)$ real and even
  - $x(t)$ real and odd $\Leftrightarrow X(j\omega)$ purely imaginary and odd

Even-Odd Decomposition

- For a real signal: $x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$

$$\boxed{\begin{aligned} x_e(t) &= \frac{1}{2}[x(t) + x(-t)] \xleftrightarrow{\mathcal{F}} \mathcal{Re}\{X(j\omega)\} \\ x_o(t) &= \frac{1}{2}[x(t) - x(-t)] \xleftrightarrow{\mathcal{F}} j \cdot \mathcal{Im}\{X(j\omega)\} \end{aligned}}$$

- **Proof:**
  - Since $x(t)$ is real, then $X(j\omega)$ is conjugate symmetric

$$\mathcal{F}\{x_e(t)\} = \frac{1}{2}[X(j\omega) + X(-j\omega)\ ] = \frac{1}{2}[X(j\omega) + X^*(j\omega)] = \mathcal{Re}\{X(j\omega)\}$$

$$\mathcal{F}\{x_o(t)\} = \frac{1}{2}[X(j\omega) - X(-j\omega)\ ] = \frac{1}{2}[X(j\omega) - X^*(j\omega)] = j \cdot \mathcal{Im}\{X(j\omega)\}$$

Example

For $a > 0$, let

$$x(t) = \begin{cases} e^{-at}, & t > 0 \\ -e^{at} & t < 0 \end{cases}$$ odd

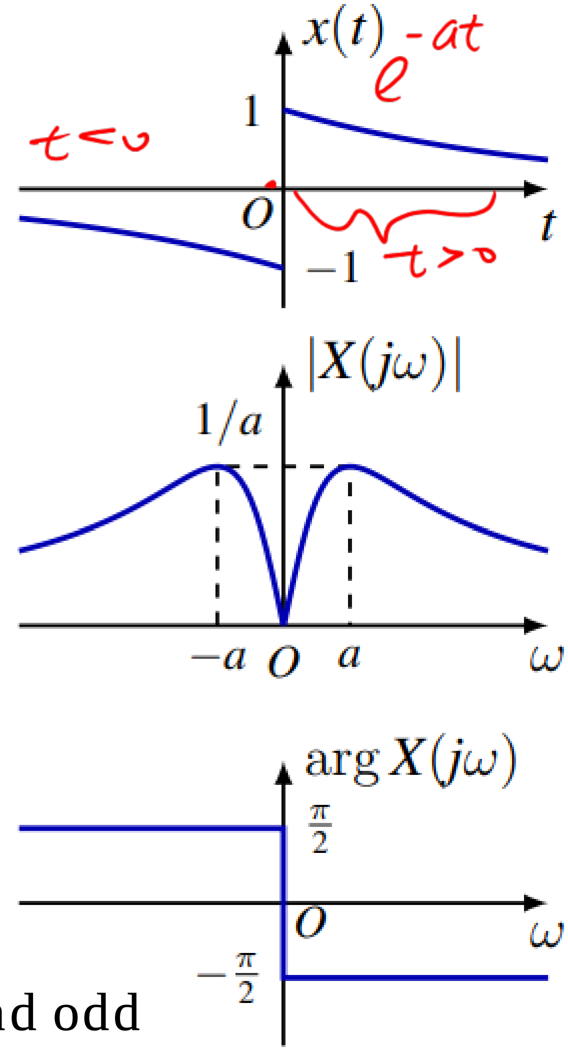
- Let $y(t) = e^{-at}u(t)$, with

$$Y(j\omega) = \frac{1}{a + j\omega}$$

- Since $x(t) = y(t) - y(-t)$

$$X(j\omega) = Y(j\omega) - Y(-j\omega)$$
$$= \frac{1}{a + j\omega} - \frac{1}{a - j\omega} = \frac{-2j\omega}{a^2 + \omega^2}$$

- $x(t)$ real and odd $\Longrightarrow$ $X(j\omega)$ purely imaginary and odd



Time and Frequency Scaling

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$\boxed{x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right), a \neq 0}$$

- **Proof:** change variables by $\tau = at$
  - For $a > 0$,

$$\int_{-\infty}^{+\infty} x(at)e^{-j\omega t}dt = \int_{-\infty}^{+\infty} x(\tau)e^{-j\frac{\omega}{a}\tau}d\left(\frac{\tau}{a}\right) = \frac{1}{a}X\left(\frac{j\omega}{a}\right)$$

  - For $a < 0$,

$$\int_{-\infty}^{+\infty} x(at)e^{-j\omega t}dt = \int_{\infty}^{-\infty} x(\tau)e^{-j\frac{\omega}{a}\tau}d\left(\frac{\tau}{a}\right) = -\frac{1}{a}X\left(\frac{j\omega}{a}\right)$$

$t = \frac{\tau}{a}$

Time and Frequency Scaling

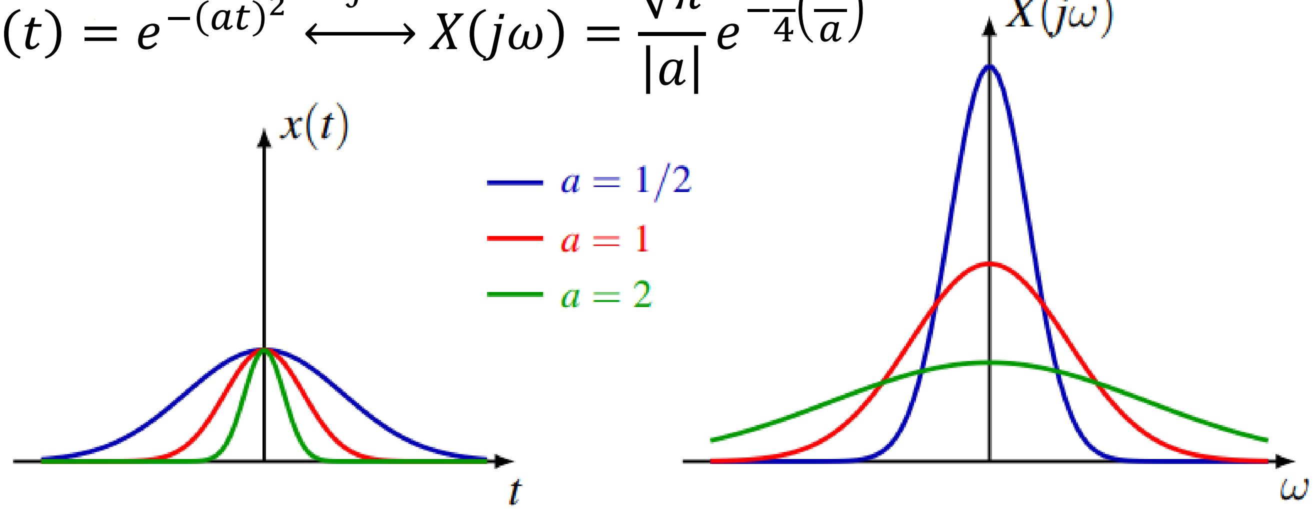
$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \qquad\qquad x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right), a \neq 0$$

compression in time ⟺ stretching in frequency

stretching in time ⟺ compression in frequency

- **Example:** faster playback an audio clip ⟹ higher pitch

$$x(t) = e^{-(at)^2} \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \frac{\sqrt{\pi}}{|a|} e^{-\frac{1}{4}\left(\frac{\omega}{a}\right)^2}$$



Exercise

- **Problem:** If we have the Fourier transform pair

$$x(t) \leftrightarrow X(j\omega),$$

determine the Fourier transform for the signal $x(at+b)$.

- **Answer:**

$$\mathcal{F}\{x(at+b)\} = \frac{1}{|a|} X(\frac{j\omega}{a}) e^{j\frac{\omega b}{a}}$$

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Differentiation

$$x'(t) \xleftrightarrow{\mathcal{F}} j\omega X(j\omega), \qquad x^{(k)}(t) \xleftrightarrow{\mathcal{F}} (j\omega)^k X(j\omega)$$

$j\omega X(j\omega)$

- **Proof:** differentiate both sides of FT synthesis equation

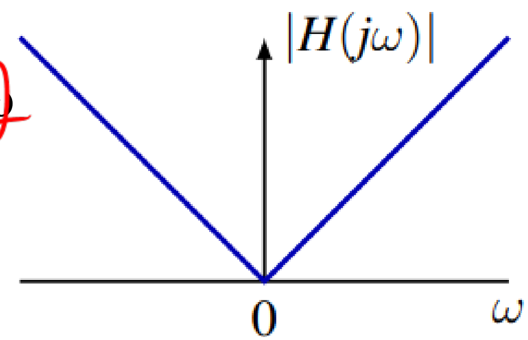
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \implies x'(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)e^{j\omega t}d\omega$$

$h(t) = \delta'(t)$

- **Example:** differentiator $y(t) = x'(t)$, with frequency response

$$H(j\omega) = \mathcal{F}\{\delta'(t)\} = \int_{-\infty}^{+\infty} \delta'(t)e^{-j\omega t}dt$$

$$= \delta(t)e^{-j\omega t}\Big|_{-\infty}^{+\infty} + j\omega\underbrace{\int_{-\infty}^{+\infty} \delta(t)e^{-j\omega t}dt}_{1} = j\omega$$



  - $Y(j\omega) = X(j\omega)H(j\omega)$
  - amplifies high frequencies
  - suppresses low frequencies
  - DC completely eliminated, cannot be completely recovered from $x'(t)$

$x'(t) = \frac{dx(t)}{dt}$

$\frac{d(c+x(t))}{dt}$

Integration

$$y(t)=\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathcal{F}}{\longleftrightarrow} Y(j\omega)=\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$$

- Since $x(t)=y'(t)$, by differentiation property

$$X(j\omega)=j\omega Y(j\omega) \implies Y(j\omega)=\frac{1}{j\omega}X(j\omega), \forall\omega\neq 0$$

- Intuitively, $y(t)$ has a DC component

$$\bar{y}=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} y(t)dt=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau\, dt$$

$$=\int_{-\infty}^{\infty} x(\tau)\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} u(t-\tau)dt\right]d\tau=\frac{1}{2}\int_{-\infty}^{\infty} x(\tau)d\tau=\frac{1}{2}X(0)$$

- Therefore, at $\omega=0$, $Y(j\omega)$ has a component $2\pi\bar{y}\delta(\omega)=\pi X(0)\delta(\omega)$

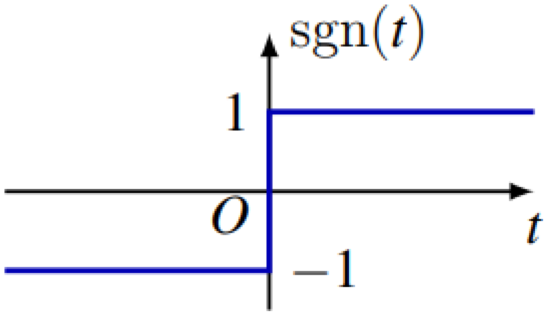
$1 \leftrightarrow 2\pi\delta(\omega)$

$\frac{1}{2}$

Integration

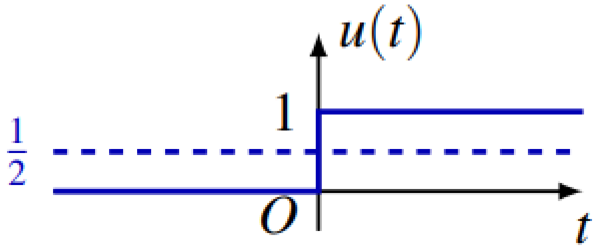
- **Sign function:**

$$\mathrm{sgn}(t)=\begin{cases}1, & t>0\\ -1, & t<0\end{cases} \xleftrightarrow{\mathcal{F}} \frac{2}{j\omega}$$



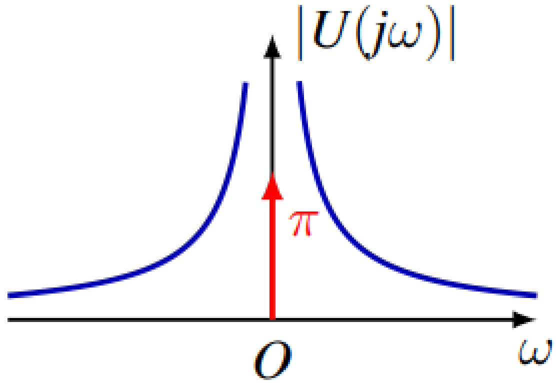
- **Constant function:**

$$1 \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega)$$



- **Unit step function:**

$$u(t)=\frac{\mathrm{sgn}(t)+1}{2} \xleftrightarrow{\mathcal{F}} U(j\omega)=\frac{1}{j\omega}+\pi\delta(\omega)$$



- **Integrator:**

$$y(t)=\int_{-\infty}^{t}x(\tau)d\tau=x(t)*u(t) \xleftrightarrow{\mathcal{F}} Y(j\omega)=X(j\omega)U(j\omega)=\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$$

Example: Triangular Pulse

$\delta(t) \xleftrightarrow{F} 1$

- **Sign function:**

$$x(t) = \begin{cases} 1 - \dfrac{2|t|}{T}, & |t| \le \dfrac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

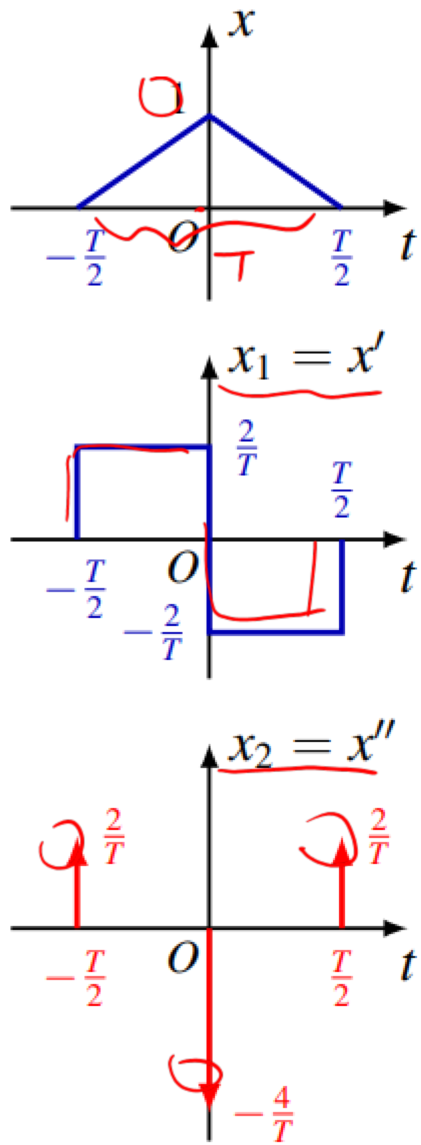
$$\cancel{X_2(j\omega)} = \frac{2}{T}\left(e^{j\frac{\omega T}{2}} - 2 + e^{j\frac{\omega T}{2}}\right) = -\frac{8}{T}\sin^2\left(\frac{\omega T}{4}\right) \quad \frac{1}{j\omega}$$

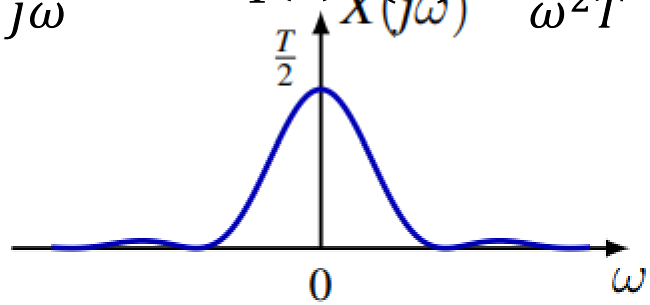
$$\cancel{X_1(j\omega)} = \frac{X_2(j\omega)}{j\omega} + \cancel{\pi X_2(0)\delta(\omega)} = -\frac{8}{j\omega T}\sin^2\left(\frac{\omega T}{4}\right) \quad \frac{1}{j\omega}$$

$\pi X_2(0) = 0$

$$\cancel{X(j\omega)} = \frac{X_1(j\omega)}{j\omega} + \cancel{\pi X_1(0)\delta(\omega)} = \frac{8}{\omega^2 T}\sin^2\left(\frac{\omega T}{4}\right)$$

$\pi X_1(0) = 0$





Calculation of $X(0)$

- **Method 1:**

$$X(0) = X(j\omega)\Big|_{\omega=0}$$

- **Method 2:**

$$X(0) = \int_{-\infty}^{+\infty} x(t)dt$$

$$= X(j\omega)\Big|_{\omega=0} = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt\Big|_{\omega=0}$$

$$= \int_{-\infty}^{\infty} x(t)dt$$

Example: $\mathrm{sgn}(t)$ Function

- **In terms of exponential function in the limit** $(a > 0)$ $a \to 0$

$$x(t) = \begin{cases} e^{-at}, & t > 0 \\ -e^{at} & t < 0 \end{cases} \longleftrightarrow X(j\omega) = \frac{-2j\omega}{a^2+\omega^2}$$

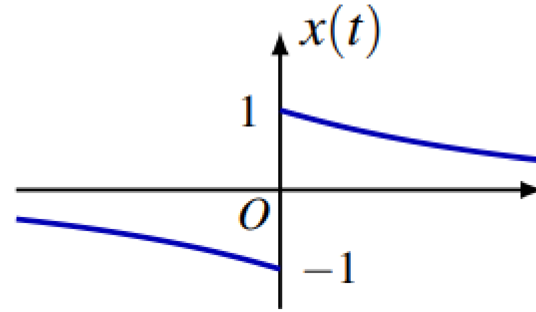
$$\mathrm{sgn}(t) = \lim_{a\to 0} x(t)$$

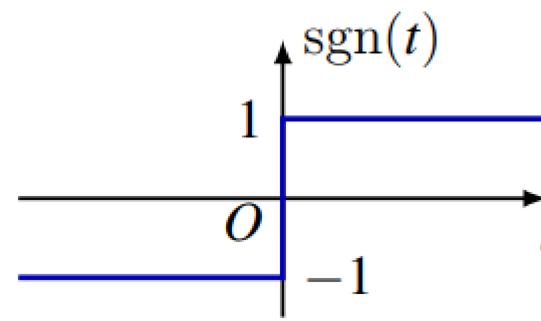
$$\mathrm{sgn}(t) \overset{\mathcal{F}}{\longleftrightarrow} \lim_{a\to 0} X(j\omega) = \frac{2}{j\omega}$$

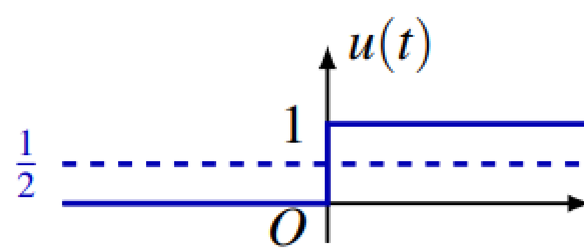
- **In terms of unit step functions**

$$\mathrm{sgn}(t) = u(t) - u(-t)$$

$$\mathrm{sgn}(t) \overset{\mathcal{F}}{\longleftrightarrow} \left[\pi\delta(\omega) + \frac{1}{j\omega}\right] - \left[\pi\delta(-\omega) + \frac{1}{-j\omega}\right] = \frac{2}{j\omega}$$







Example: sgn(t) Function

- **Exploiting integration property**

$$\mathrm{sgn}(t) = 2u(t) - 1 \triangleq x(t)$$

$$\mathrm{sgn}'(t) = 2\delta(t) \triangleq x_1(t)$$

$$x_1(t) = 2\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(j\omega) = 2$$

$$\mathrm{sgn}(t) = x(t) = x(-\infty) + \int_{-\infty}^{t} x_1(t)dt$$

$$\Longrightarrow \mathcal{F}\{\mathrm{sgn}(t)\} = X(j\omega) = 2\pi x(-\infty)\delta(\omega) + \frac{X_1(j\omega)}{j\omega} + \pi X_1(0)\delta(\omega)$$

$$= 2\pi \cdot (-1) \cdot \delta(\omega) + \frac{2}{j\omega} + \pi \cdot 2\delta(\omega) = \frac{2}{j\omega}$$

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Duality

- Both **time and frequency** functions are continuous and aperiodic in general, **identical form except** for
  - different signs in exponent of complex exponential
  - constant factor $1/2\pi$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

- Suppose two functions are related by

$$f(r) = \int_{-\infty}^{\infty} g(\tau)e^{-jr\tau}\,d\tau$$

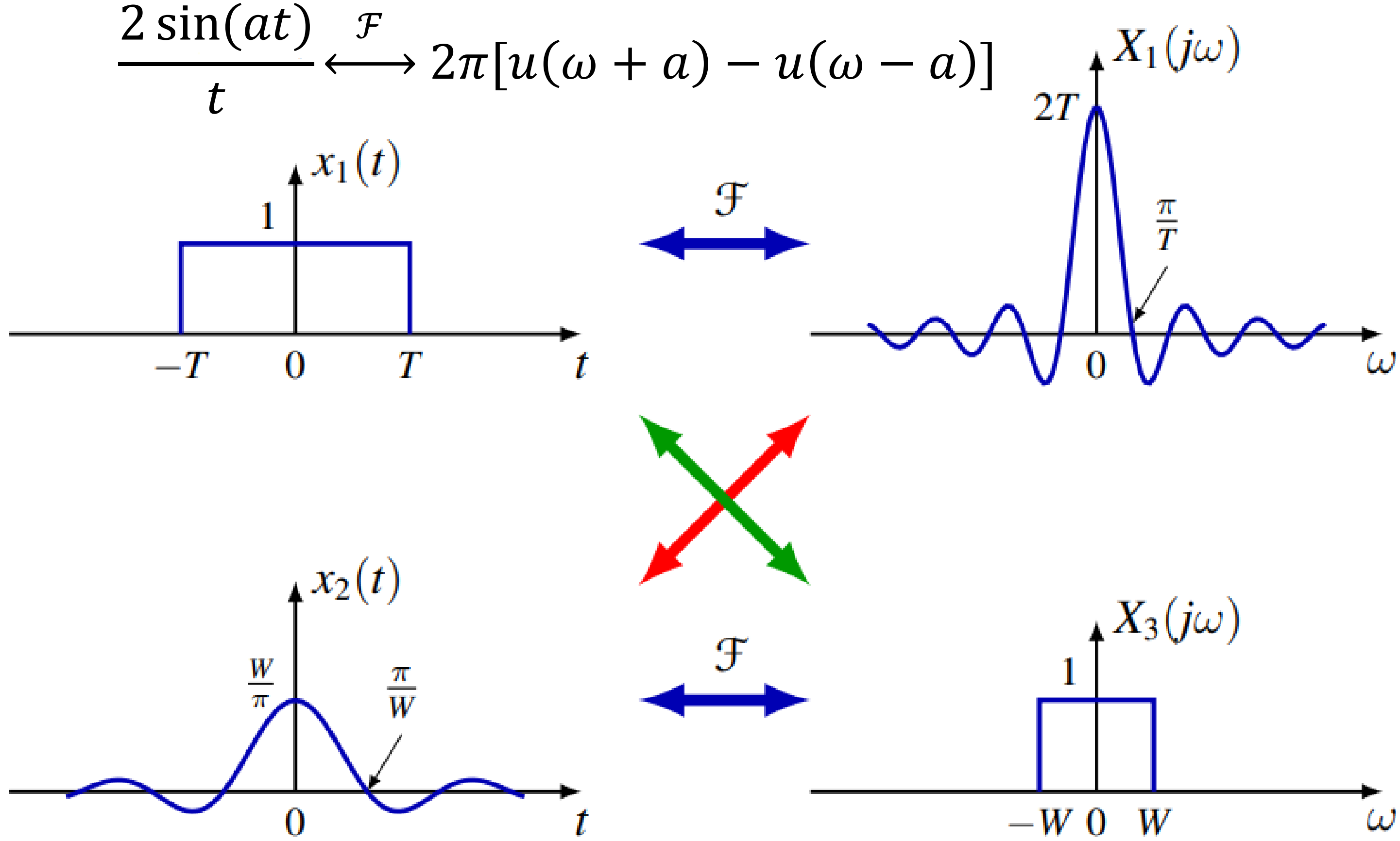
  - Let $\tau = t$, and $r = \omega$, $\Longrightarrow\ g(t)\overset{\mathcal{F}}{\longleftrightarrow}f(\omega)$
  - Let $\tau = -\omega$, and $r = t$,$\Longrightarrow f(t)\overset{\mathcal{F}}{\longleftrightarrow}2\pi g(-\omega)$

- **Duality:** $\boxed{x(t)\overset{\mathcal{F}}{\longleftrightarrow}X(j\omega)\ \Leftrightarrow\ X(t)\overset{\mathcal{F}}{\longleftrightarrow}2\pi x(-j\omega)}$

Duality

$$u(t+a) - u(t-a) \overset{\mathcal{F}}{\longleftrightarrow} \frac{2\sin(a\omega)}{\omega}$$

$$\frac{2\sin(at)}{t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi[u(\omega+a) - u(\omega-a)]$$



Duality

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \iff X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi x(-j\omega)$$

Example: $x(t) = \frac{1}{\pi t}$

- **Known**

$$\mathrm{sgn}(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{2}{j\omega}$$

- **By duality**
  - switch LHS and RHS, switch ω and t
  - substitute $\omega \to -\omega$ or $t \to -t$ (but **not** both)
  - multiply RHS by 2π

$$\frac{1}{2\pi}\quad -\frac{2}{jt} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi \cdot \mathrm{sgn}(\omega) \quad \frac{1}{2\pi}$$

- **By linearity**

$$\frac{1}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} -j \cdot \mathrm{sgn}(\omega)$$

Duality

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \;\Leftrightarrow\; X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi x(-j\omega)$$

Example: $x(t) = \frac{1}{1+t^2}$

- **Known**

$$e^{-|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2}{1+\omega^2}$$

- **By duality**
  - switch LHS and RHS, switch ω and t
  - substitute $\omega \to -\omega$ or $t \to -t$ (but **not** both)
  - multiply RHS by 2π

$$\frac{1}{2}\quad \frac{2}{1+t^2(-t)} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi \cdot e^{-|\omega|} \quad \frac{1}{2}$$

- **By linearity**

$$\frac{1}{1+t^2} \overset{\mathcal{F}}{\longleftrightarrow} \pi \cdot e^{-|\omega|}$$

Duality

$\dot{x}(t) \leftrightarrow j\omega X(j\omega)$

- **Differentiation in frequency**

$$-jtx(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{dX(j\omega)}{d\omega}, \quad tx(t) \overset{\mathcal{F}}{\longleftrightarrow} j\frac{dX(j\omega)}{d\omega}$$

j j

- **Proof:**

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \Longrightarrow \frac{dX(j\omega)}{d\omega} = \int_{-\infty}^{\infty} (-jt)x(t)e^{-j\omega t}\,dt$$

- **Example:** Unit ramp function $u_{-2}(t) = \int_{-\infty}^{t} u(\tau)d\tau = tu(t)$

$$\mathcal{F}\{u_{-2}(t)\} = j\mathcal{F}\{-jtu(t)\} = j\frac{dU(j\omega)}{d\omega} = -\frac{1}{\omega^2} + j\pi\delta'(\omega)$$

- **Integration in frequency**

$x(0)=0$

$\frac{1}{t}x(t) \leftrightarrow j\int_{-\infty}^{\omega} X(\eta)d\eta$

$$-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\omega} X(\eta)d\eta$$

Example

- **Problem:** $x(t) = te^{-2t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} ?$

- **Solution:**

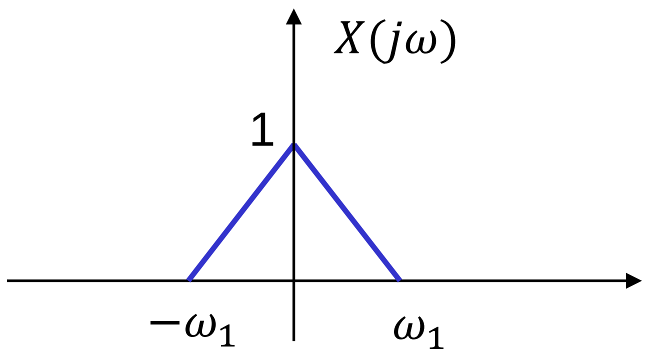
$$e^{-2t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2+j\omega}$$

$$te^{-2t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}\left(\frac{1}{2+j\omega}\right) = \frac{1}{(2+j\omega)^2}$$

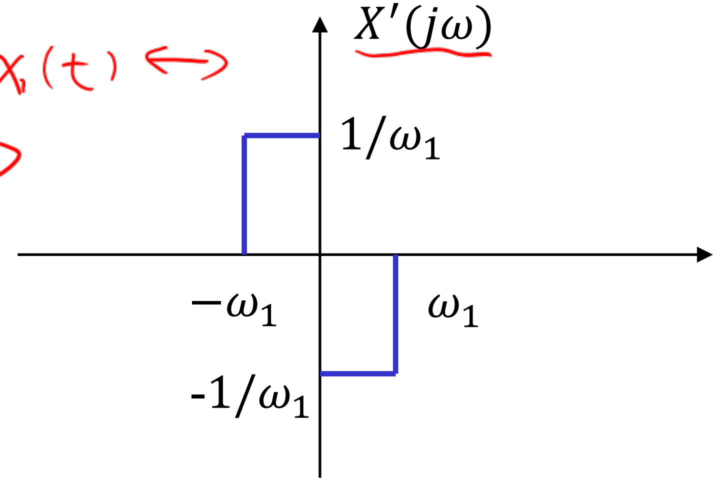
- **Exercise:** $x(t) = te^{-2t}u(t-1) \leftrightarrow ?$

Exercise

- Determine $x(t)$ according to $X(j\omega)$



$\Rightarrow$ $x_1(t) \longleftrightarrow$



- **Hints:** exploit integration property in frequency domain

$$x_1(t) \overset{\mathcal{F}}{\longleftrightarrow} X'(j\omega) \;\Longrightarrow\; x(t) = \frac{x_1(t)}{-jt} + \pi x_1(0)\delta(t)$$

where $$x_1(0) = \int X'(j\omega)\, d\omega = 0$$

Parseval's Relation

- For a Fourier transform pair $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$

$$\boxed{\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega = \int_{-\infty}^{\infty} |X(j\omega)|^2 \frac{d\omega}{2\pi}}$$

- **Note:** ω is angular frequency and $f = \omega/2\pi$ is frequency
- **Interpretation: Energy conservation**
  - $|x(t)|^2$: power, or energy per unit time (in second)
  - $|X(j\omega)|^2$: energy per unit frequency (in Hertz), called **energy-density spectrum**
- $\lim_{T\to\infty} \frac{1}{T} \int_T |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|X(j\omega)|^2}{T} d\omega$
  - $\lim_{T\to\infty} \frac{|X(j\omega)|^2}{T}$: called **power-density spectrum**

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Parseval's Relation

- For a Fourier transform pair $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega = \int_{-\infty}^{\infty} |X(j\omega)|^2 \frac{d\omega}{2\pi}$$

- **Proof:**

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} x(t)x^*(t)\, dt = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega\right]^* dt$$

$$= \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\, e^{-j\omega t} d\omega\right] dt$$

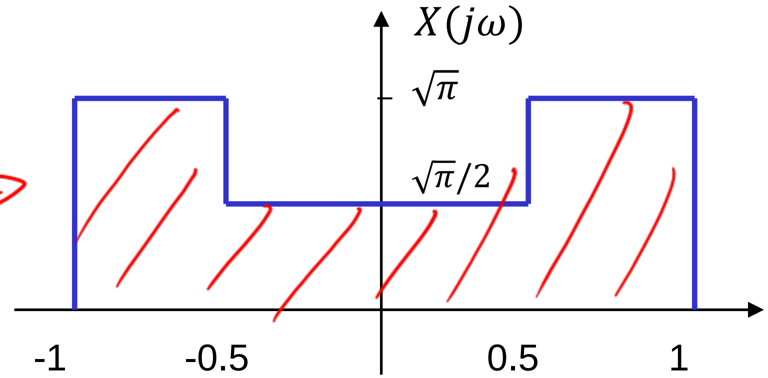
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)X(j\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega$$

Parseval's Relation

- **Example:** determine the following time domain expressions:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt, \text{ and } D = x'(t)\Big|_{t=0}$$



- **Solution:**

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega = \frac{1}{2\pi}\cdot\frac{5\pi}{4} = \frac{5}{8}$$

$$g(t) = x'(t) \overset{\mathcal{F}}{\longleftrightarrow} G(j\omega) = j\omega X(j\omega)$$

$$\Rightarrow D = g(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)\ d\omega = 0$$

$x'(t)\big|_{t=0}$

Parseval's Relation

- **Example:**

$$\frac{\sin(Wt)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} u(\omega + W) - u(\omega - W)$$

- **By Parseval's relation**

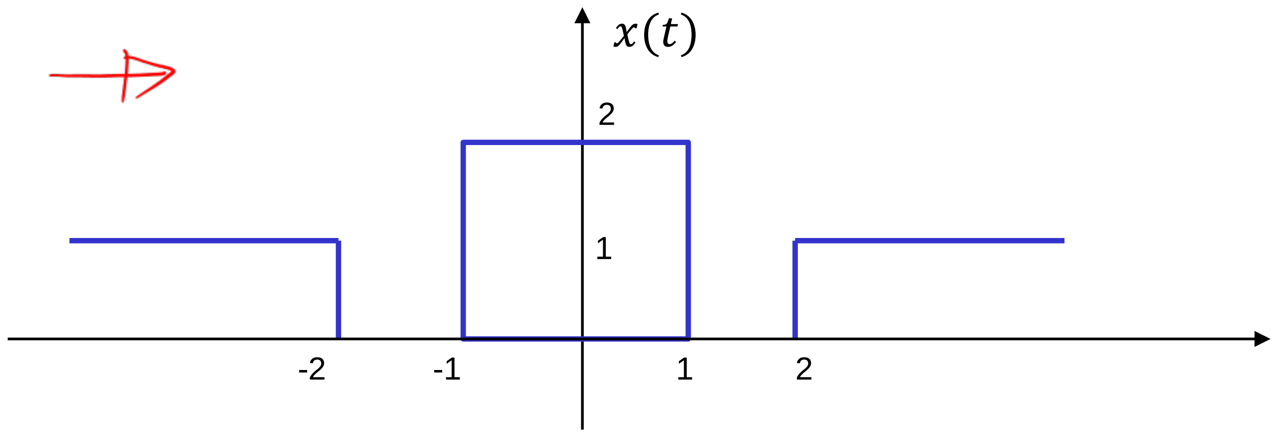
$$\int_{-\infty}^{\infty} \frac{\sin^2(Wt)}{\pi^2 t^2} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |u(\omega + W) - u(\omega - W)|^2 \, d\omega$$

$$= \frac{1}{2\pi} \int_{-W}^{W} 1 \, d\omega$$

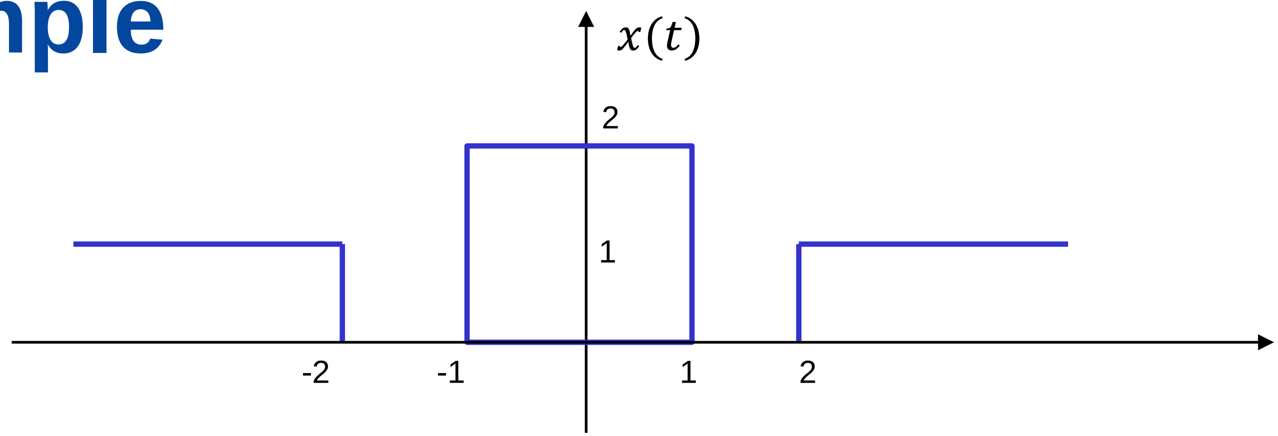
$$= \frac{W}{\pi}$$

Example

- Use **Fourier transform of typical signals** and **Fourier transform properties** to determine the Fourier transform of the following signal

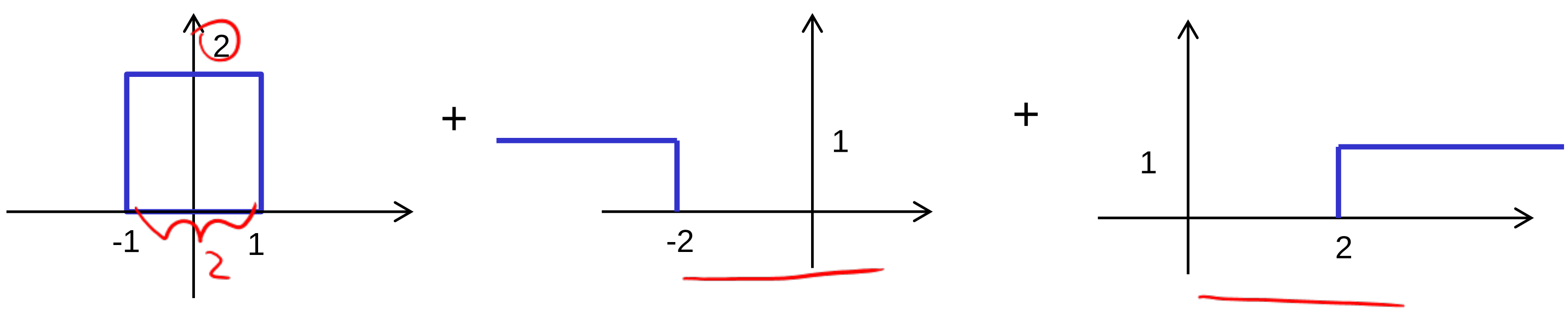


Example



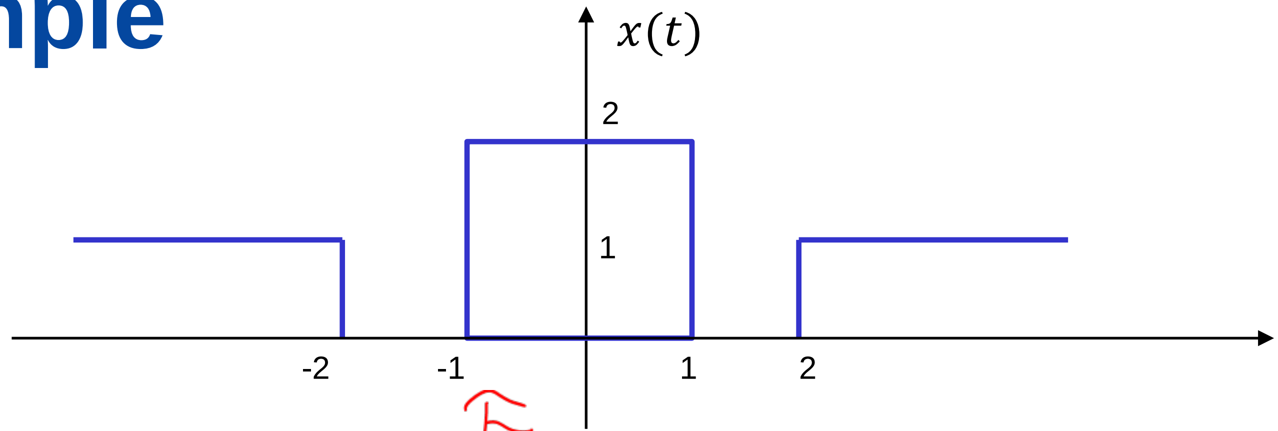
- **Solution 1:**

$$x(t) = 2g_2(t) + u(-t-2) + u(t-2)$$

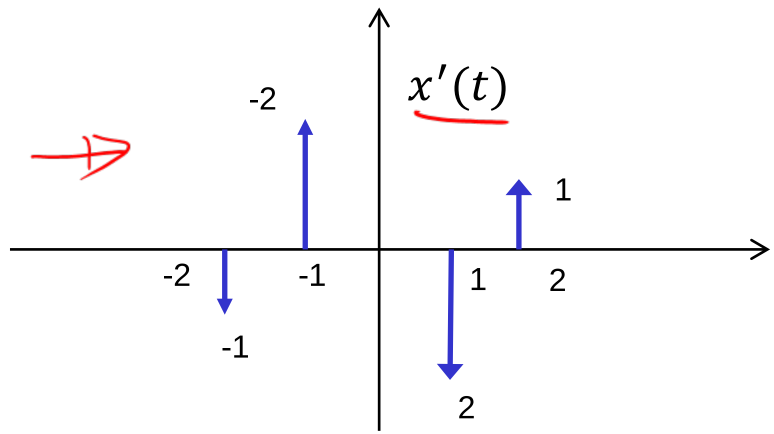


  - $g_\tau(t)$ is the rectangular pulse with width of τ and unit magnitude

Example



$\delta(t) \xleftrightarrow{\mathcal{F}} 1$



- **Solution 2:**
  - Assume

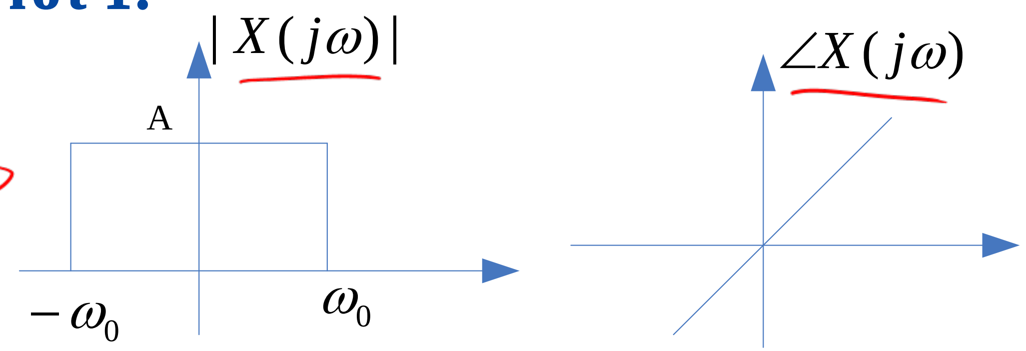
$$x_1(t) = x'(t) \xleftrightarrow{\mathcal{F}} X_1(j\omega)$$

  - Then

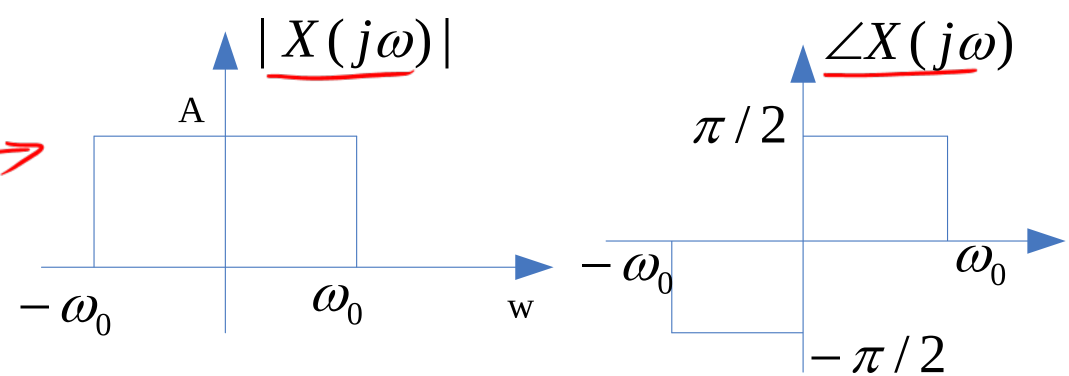
$$X(j\omega) = 2\pi x(-\infty)\delta(\omega) + \frac{X_1(j\omega)}{j\omega} + \pi X_1(0)\delta(\omega)$$

Exercise

- Determine the inverse Fourier transform
  - **Plot 1:**



  - **Plot 2:**



Note: **different phase spectrum**

Exercise

- Determine the Fourier transform of the following signals
  - $x(t) = e^{-2t}u(t)$
  - $x(t) = e^{-2(t-1)}u(t)$
  - $x(t) = e^{-2t}u(t-1)$

$x(t) \to h(t) \to y(t)=x(t)*h(t)$

$Y(j\omega)=X(j\omega)H(j\omega)$

Convolution Property

$$\boxed{y(t)=x(t)*h(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)=X(j\omega)H(j\omega)}$$

convolution in time ⟺ multiplication in frequency

Proof:

$$Y(j\omega)=\int_{-\infty}^{+\infty} y(t)e^{-j\omega t}dt=\int_{-\infty}^{+\infty} x(t)*h(t)e^{-j\omega t}dt$$

$$=\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\,d\tau\right]e^{-j\omega t}dt$$

$$=\int_{-\infty}^{+\infty} x(\tau)\left[\int_{-\infty}^{+\infty} h(t-\tau)\,e^{-j\omega t}dt\right]d\tau$$

$$=\int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}\boxed{H(j\omega)}d\tau=H(j\omega)\int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}d\tau=H(j\omega)X(j\omega)$$

Convolution Property

- For an LTI system with
  - impulse response $h(t)$
  - frequency response $H(j\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t} dt = \mathcal{F}\{h(t)\}$
  - $e^{j\omega t}$ is eigenfunction associated with eigenvalue $H(j\omega)$

$$x(t) = e^{j\omega t} \;\rightarrow\; y(t) = \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)} d\tau = H(j\omega) e^{j\omega t}$$

- Input $x(t)$ as a linear combination of complex exponentials

$X_T(t)$ $\quad T \to \infty$

$$\Rightarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega = \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0$$

$H(j\omega)$

$$\Rightarrow y(t) = \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) H(jk\omega_0) e^{jk\omega_0 t} \omega_0$$

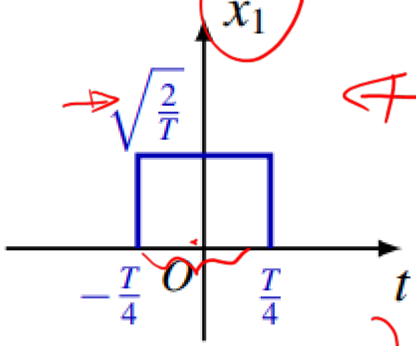
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) H(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(j\omega) e^{j\omega t} d\omega$$

Example

- **Rectangular pulse:**

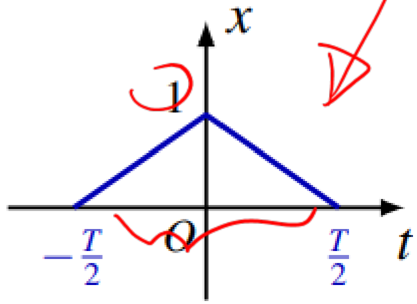
$$x_1(t) = \sqrt{2/T}\left[u\left(t+\frac{T}{4}\right) - u(t-\frac{T}{4})\right]$$

$$X_1(j\omega) = \sqrt{2/T}\cdot\frac{2\sin(\omega T/4)}{\omega}$$

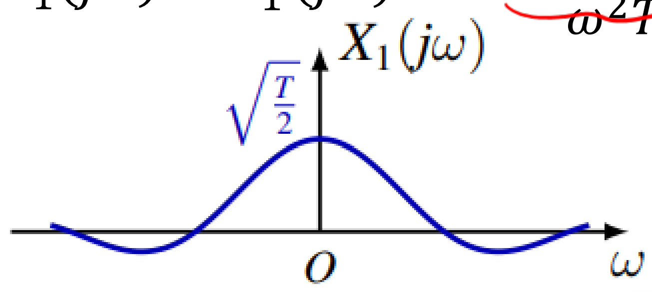


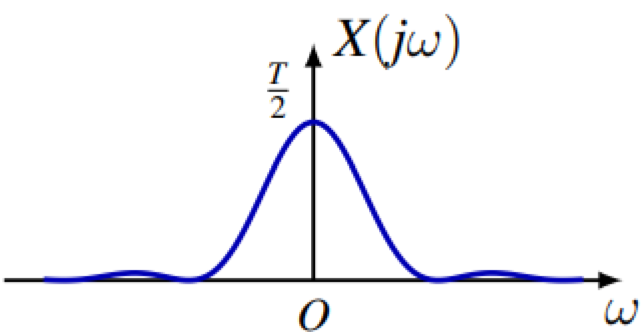
- **Triangular pulse:**

$$x(t) = \left(1-\frac{2|t|}{T}\right)\left[u\left(t+\frac{T}{2}\right) - u(t-\frac{T}{2})\right]$$



$$\cancel{X_1(j\omega)} = X_1^2(j\omega) = \frac{8\sin^2(\omega T/4)}{\omega^2 T}$$





Frequency Response of LTI Systems

- Fully characterized by **impulse response** $h(t)$

$$y(t) = x(t) * h(t)$$

- Also fully characterized by **frequency response** $H(j\omega) = \mathcal{F}\{h(t)\}$, if $H(j\omega)$ is well defined
  - BIBO stable system: $\int_{-\infty}^{+\infty} |h(t)| dt < \infty$
  - other systems: e.g., identity $h(t) = \delta(t)$, differentiator $h(t) = \delta'(t)$

1 jω

- **Convolution property** implies
  - instead of computing $x(t) * h(t)$ in **time domain**, we can analyze a system in **frequency domain**

$$y(t) = x(t) * h(t)$$

$$\Downarrow$$

F{h(t)}

$$Y(j\omega) = X(j\omega) \cdot H(j\omega), \qquad H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = |H(j\omega)| e^{j\angle H(j\omega)}$$

Frequency Response of LTI Systems

- **Some typical systems:**

F

	$h(t)$	$H(j\omega)$
id	$\delta(t)$	1
τ_{t_0}	$\delta(t-t_0)$	$e^{-j\omega t_0}$
$\dfrac{d}{dt}$	$\delta'(t)$	$j\omega$
$\int_{-\infty}^{t}$	$u(t)$	$\dfrac{1}{j\omega}+\pi\delta(\omega)$
ideal lowpass	$\dfrac{\sin(\omega_c t)}{\pi t}$	$u(\omega+\omega_c)-u(\omega-\omega_c)$
1st order lowpass	$\frac{1}{\tau}e^{-t/\tau}u(t)$	$\dfrac{1}{1+j\tau\omega}$

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Examples

- **Example:** Differentiation property

$$y(t) = x'(t) = x(t) * \delta'(t)$$

$$Y(j\omega) = X(j\omega) \cdot \mathcal{F}\{\delta'(t)\} = j\omega \cdot X(j\omega)$$

$\delta(t) \Leftrightarrow 1$

$H(j\omega) = j\omega$

- **Example.** Integration property

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t)$$

$$Y(j\omega) = X(j\omega) \cdot U(j\omega) = X(j\omega)\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$$= \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

Example

h(t)

- **Unit ramp function:** $u_{-2}(t) = u(t) * u(t) = tu(t)$

- **Convolution property** suggests

$$\mathcal{F}\{u_{-2}(t)\} = U^2(j\omega) = \left[\frac{1}{j\omega} + \pi\delta(\omega)\right]^2 = -\frac{1}{\omega^2} + \frac{2\pi}{j\omega}\delta(\omega) + \pi^2\delta(\omega)\delta(\omega)$$

  - But $\frac{\delta(\omega)}{j\omega}$ and $\delta(\omega)\delta(\omega)$ **not well-defined!**
  - Convolution property **not applicable** here
  - **Rule of thumb:** applicable when formula is **well-defined**

- Known from the ~~differentiation in frequency property~~

U(jω)

$$\mathcal{F}\{u_{-2}(t)\} = j\mathcal{F}\{-jtu(t)\} = j\frac{dU(j\omega)}{d\omega} = -\frac{1}{\omega^2} + j\pi\delta'(\omega)$$

F{tu(t)}

Example

- Determine the Response of an LTI system with impulse response $h(t) = e^{-at}u(t)$, to the input $x(t) = e^{-bt}u(t)$, where $a, b > 0$

- **Method 1: convolution** in time domain $y(t) = x(t) * h(t)$

- **Method 2:** solving the **differential equation with initial rest condition** $y'(t) + ay(t) = e^{-bt}u(t)$

  Ae^{-at} $\quad \delta(t) \quad h(0_+) = 1$

  $A = 1$

- **Method 3: Fourier transform**

$$H(j\omega) = \frac{1}{a + j\omega}, X(j\omega) = \frac{1}{b + j\omega} \Rightarrow Y(j\omega) = \frac{1}{(a + j\omega)(b + j\omega)}$$

$\frac{1}{(a+j\omega)^2}$

  - If $a \neq b$, $Y(j\omega) = \frac{1}{b-a}\left(\frac{1}{a+j\omega} - \frac{1}{b+j\omega}\right) \Rightarrow y(t) = \frac{1}{b-a}\left(e^{-at} - e^{-bt}\right)u(t)$ t
  - If $a = b$, $Y(j\omega) = j\frac{d}{d\omega}\left(\frac{1}{a+j\omega}\right) \Rightarrow y(t) = t \cdot \mathcal{F}^{-1}\{\frac{1}{a+j\omega}\} = te^{-at}u(t)$

Example

- Determine the Response of an LTI system with impulse response $h(t) = e^{-at}u(t)$, to the input $x(t) = \cos(\omega_0 t)$, where $a > 0$
- **Frequency response:**

$$H(j\omega) = \frac{1}{a + j\omega} = \frac{a - j\omega}{a^2 + \omega^2} = \frac{1}{\sqrt{a^2 + \omega^2}} e^{-j \cdot \arctan(\omega/a)}$$

- **Method 1:** using **eigenfunction property**

$H(-j\omega_0) = H^*(j\omega_0)$

$$x(t) = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$$

$$y(t) = \frac{1}{2} H(j\omega_0) e^{j\omega_0 t} + \frac{1}{2} H(-j\omega_0) e^{-j\omega_0 t} = \mathcal{Re}\{H(j\omega_0) e^{j\omega_0 t}\}$$

$$= \frac{a\cos(\omega_0 t) + \omega_0 \sin(\omega_0 t)}{a^2 + \omega_0^2} = \left(1/\sqrt{a^2 + \omega_0^2}\right) \cos\left[\omega_0 t - \arctan\left(\frac{\omega}{a}\right)\right]$$

Example

- Determine the Response of an LTI system with impulse response $h(t) = e^{-at}u(t)$, to the input $x(t) = \cos(\omega_0 t)$, where $a > 0$

- **Frequency response:**

$$H(j\omega) = \frac{1}{a + j\omega} = \frac{a - j\omega}{a^2 + \omega^2} = \frac{1}{\sqrt{a^2 + \omega^2}} e^{-j\cdot\arctan(\omega/a)}$$

$1 \leftrightarrow \delta(\omega)$

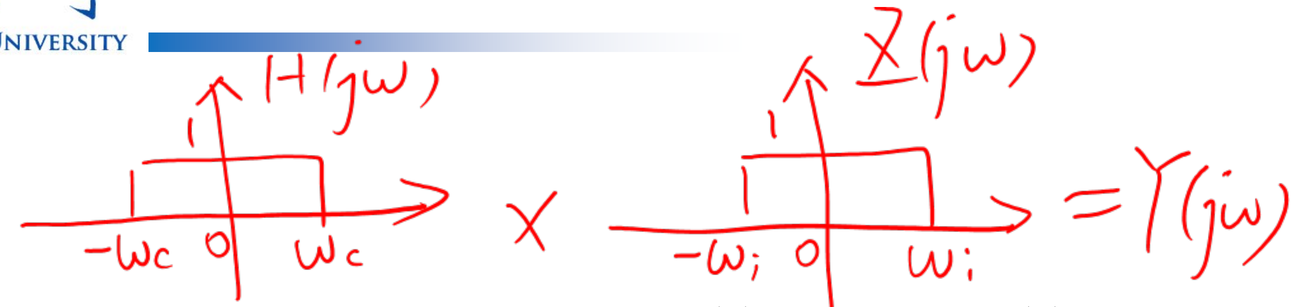
- **Method 2:** using **Fourier transform**

$$x(t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t} \Longrightarrow X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$Y(j\omega) = X(j\omega)H(j\omega) = \pi H(j\omega_0)\delta(\omega - \omega_0) + \pi H(-j\omega_0)\delta(\omega + \omega_0)$$

$$\Longrightarrow y(t) = \frac{1}{2}H(j\omega_0)e^{j\omega_0 t} + \frac{1}{2}H(-j\omega_0)e^{-j\omega_0 t}$$

Example



- Ideal lowpass filter with impulse response $h(t)$ to input $x(t)$

$$h(t) = \frac{\sin(\omega_c t)}{\pi t}, \qquad x(t) = \frac{\sin(\omega_i t)}{\pi t}$$

- **Fourier transform:**

$$X(j\omega) = u(\omega + \omega_i) - u(\omega - \omega_i)$$

$$H(j\omega) = u(\omega + \omega_c) - u(\omega - \omega_c)$$

- Use $Y(j\omega) = X(j\omega)H(j\omega)$

$$Y(j\omega) = \begin{cases} X(j\omega), & \text{if } \omega_i \le \omega_c \\ H(j\omega), & \text{if } \omega_i > \omega_c \end{cases} \Longrightarrow y(t) = \begin{cases} x(t), & \text{if } \omega_i \le \omega_c \\ h(t), & \text{if } \omega_i > \omega_c \end{cases}$$

- **Note:** Con~~volution of two $sinc()$ functions~~ is a~~nother $sinc()$ function~~

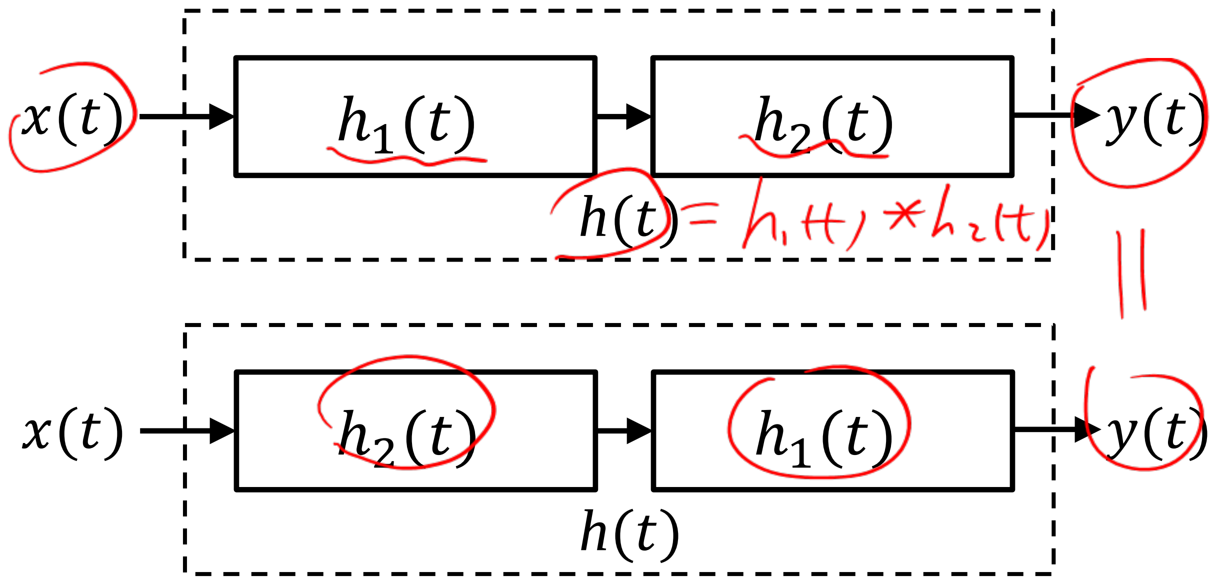
System Connections: Cascade

$$\boxed{y = (x * h_1) * h_2 = x * (h_1 * h_2) = (x * h_2) * h_1}$$

$h(t) = h_1(t) * h_2(t)$

$\updownarrow$ commutative

$h(t) = h_2(t) * h_1(t)$



Note: Order of processing usually not important for LTI systems

System Connections: Cascade

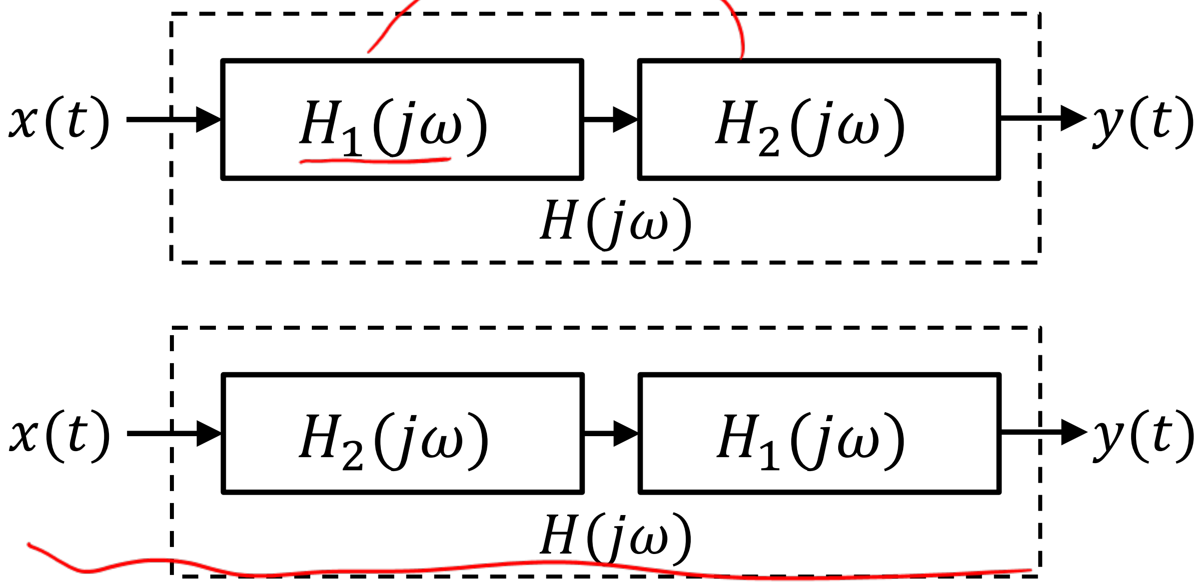
$$Y = (X \cdot H_1) \cdot H_2 = X \cdot (H_1 \cdot H_2) = (X \cdot H_2) \cdot H_1$$

$h(t) = h_1(t) * h_2(t)$

$H(j\omega) = H_1(j\omega)H_2(j\omega)$

commutative

$H(j\omega) = H_2(j\omega)H_1(j\omega)$



Note: Order of processing usually not important for LTI systems

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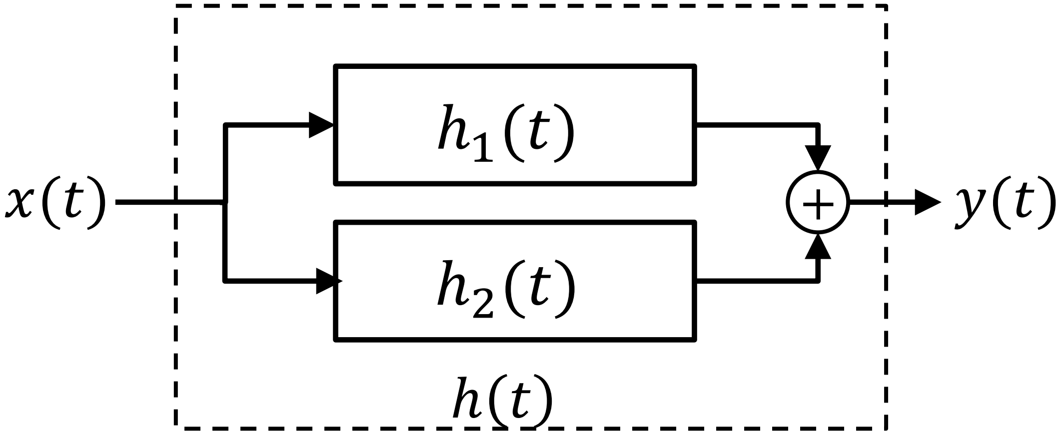
System Connections: Parallel

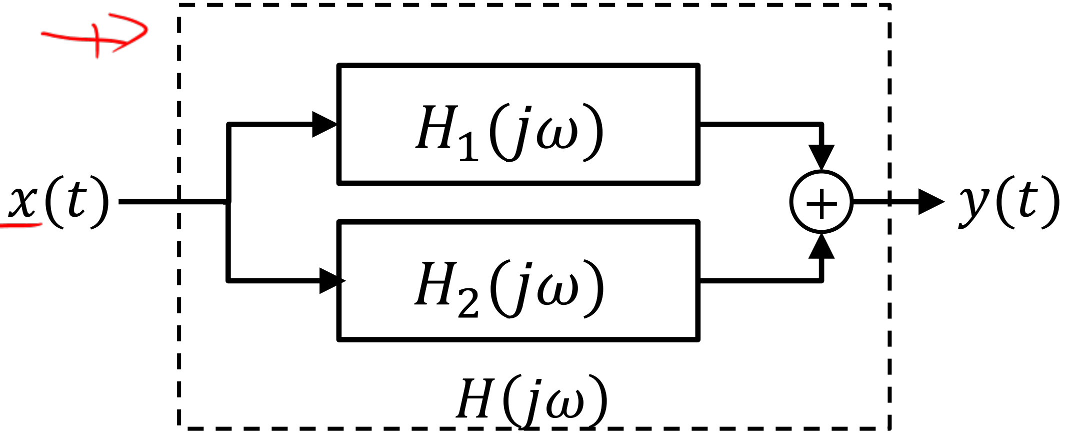
- **Time domain**

$$h(t) = h_1(t) + h_2(t)$$



- **Frequency domain**

$$H(j\omega) = H_1(j\omega) + H_2(j\omega)$$

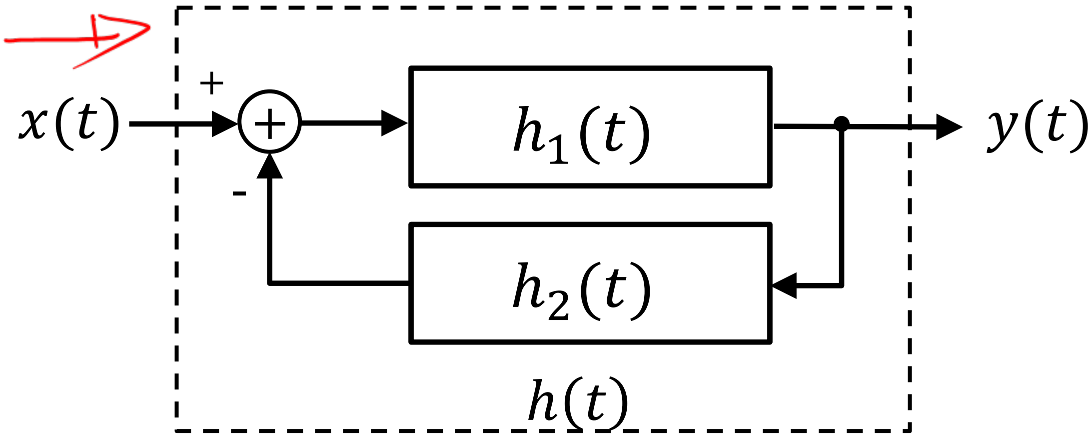


System Connections: Feedback

- **Time domain**

$$y = h_1 * (x - h_2 * y)$$

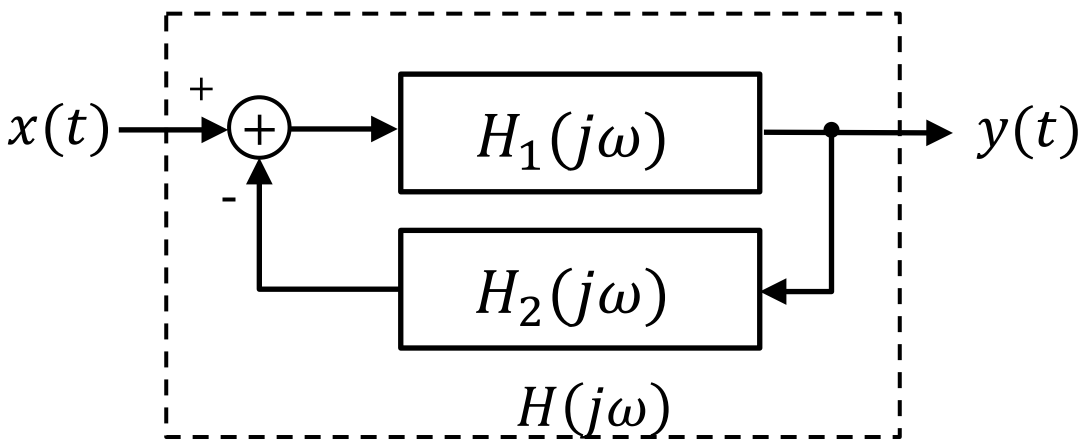
$$h(t) = ?$$



- **Frequency domain**

$$Y = H_1 X - H_1 H_2 Y$$

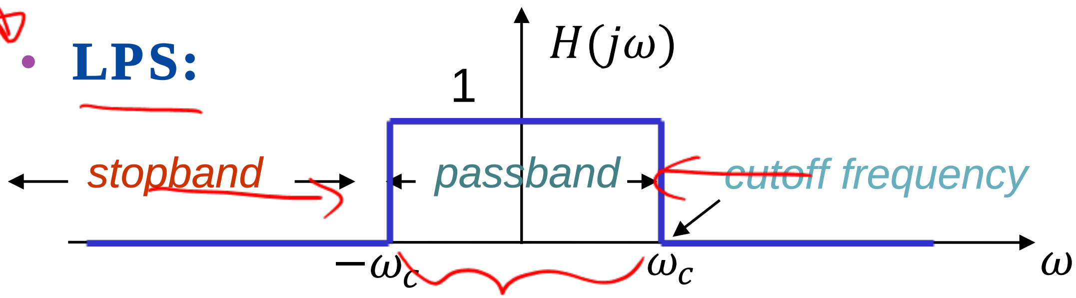
$$H = \frac{Y}{X} = \frac{H_1}{1 + H_1 H_2}$$



Filtering

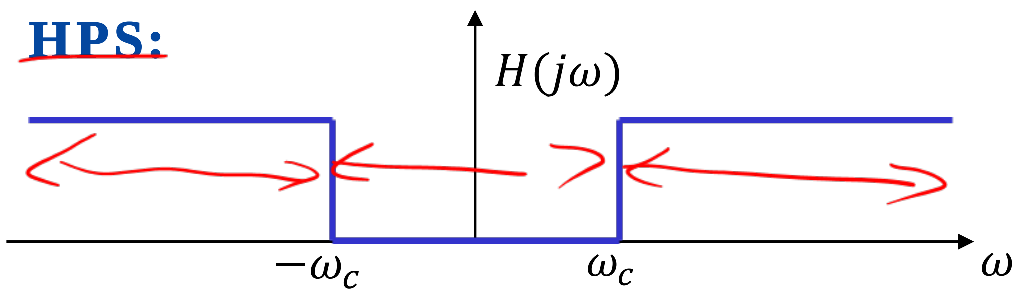
- changes relative amplitudes of frequency components, or eliminates some frequency components entirely

- **LPS:**



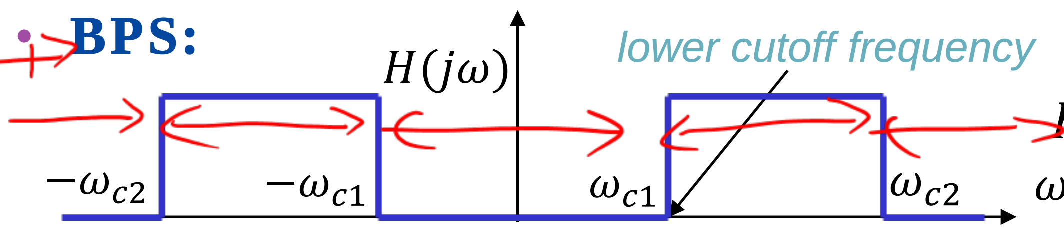
$$H(j\omega)=\begin{cases}1, & |\omega|\le\omega_c\\ 0, & |\omega|>\omega_c\end{cases}$$

- **HPS:**



$$H(j\omega)=\begin{cases}1, & |\omega|\ge\omega_c\\ 0, & |\omega|<\omega_c\end{cases}$$

- **BPS:**



$$H(j\omega)=\begin{cases}1, & \omega_{c1}<|\omega|<\omega_{c2}\\ 0, & \text{otherwise}\end{cases}$$

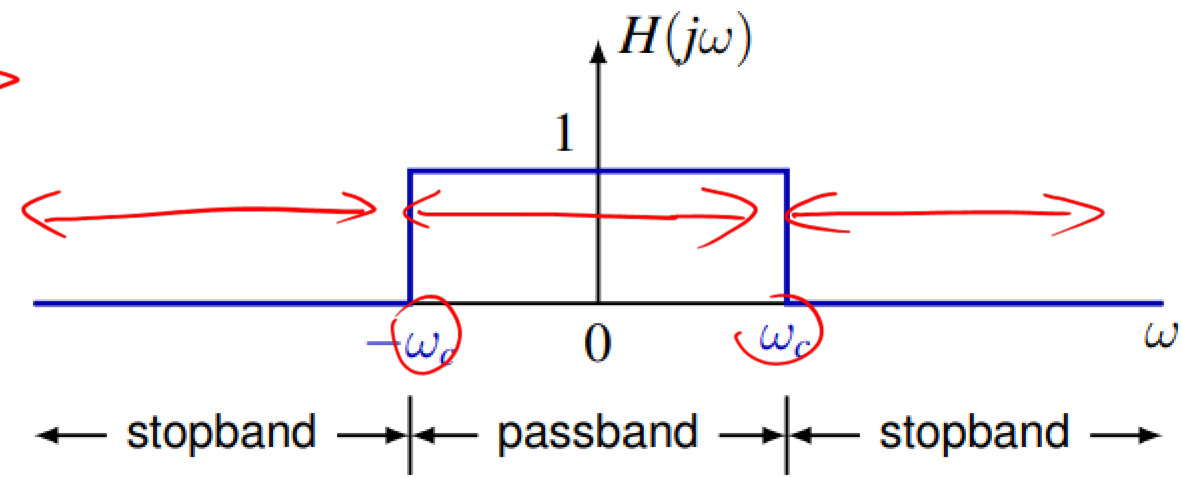
Ideal Zero Phase LPF

- **Ideal lowpass filter** (unit magnitude, zero phase)

$$h(t) = \frac{\sin(\omega_c t)}{\pi t}, \qquad H(j\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \text{otherwise} \end{cases}$$

$h(t) \neq 0,\ t<0 \Rightarrow$ non causal

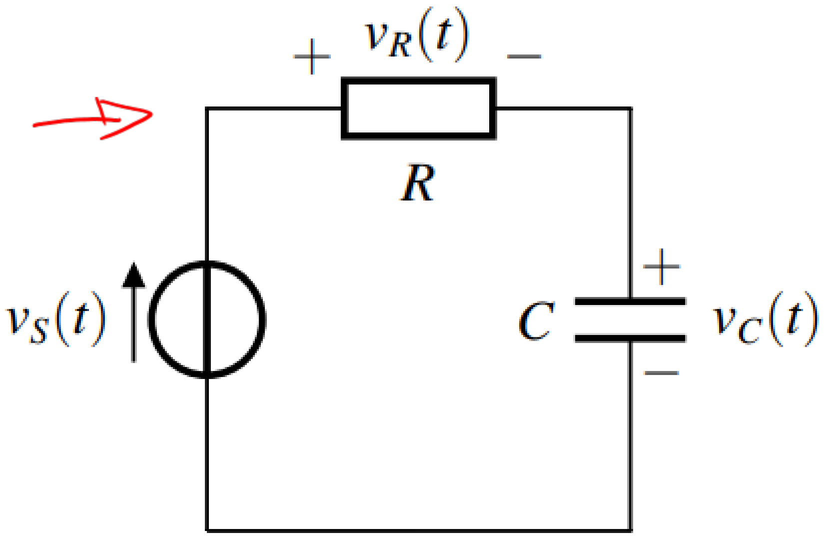
- $\boldsymbol{\omega_c}$: **cutoff frequency**



Simple RC Lowpass Filter

- **Differential Equation**

$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$



- Input $v_s(t)$, output $v_c(t)$

- Taking Fourier transform

$$j\omega RC \cdot V_c(j\omega) + V_c(j\omega) = V_s(j\omega)$$

- **Frequency response:**

$$H(j\omega) = \frac{V_c(j\omega)}{V_s(j\omega)} = \frac{1}{1 + j\omega RC}$$

Simple RC Lowpass Filter

- **Frequency response**

$$H(j\omega) = \frac{1}{1 + RCj\omega}$$

$$|H(j\omega)| = \frac{1}{\sqrt{1 + (RC\omega)^2}}$$

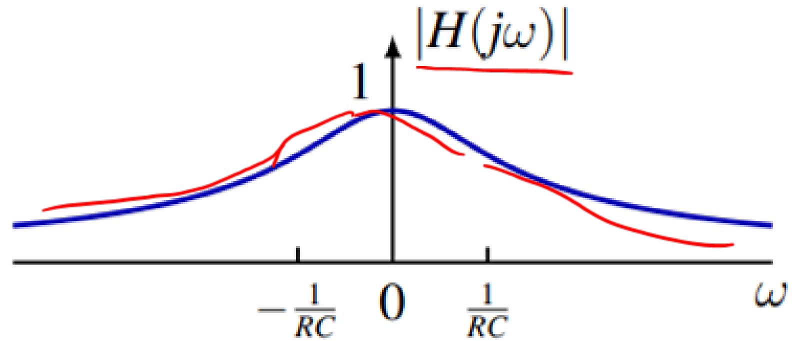
$$\arg H(j\omega) = -\arctan(RC\omega)$$

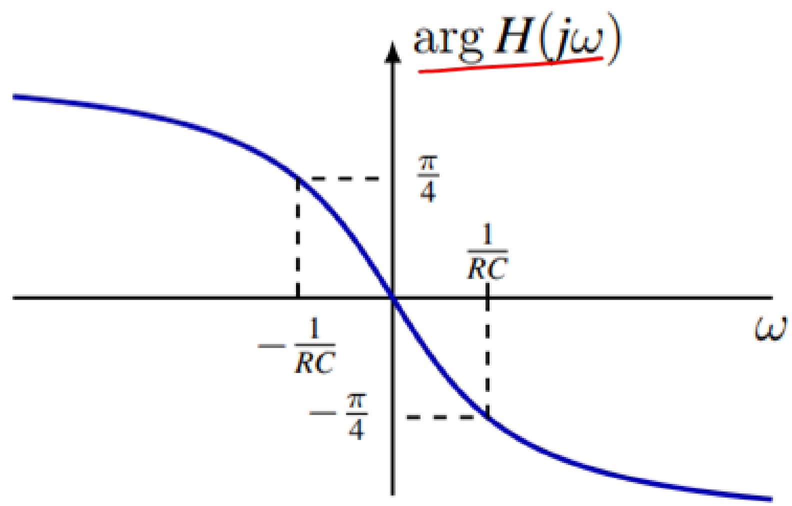
- Impulse response

causal

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

- **Nonideal lowpass filter**
  - passes lower frequencies
  - attenuates higher frequencies



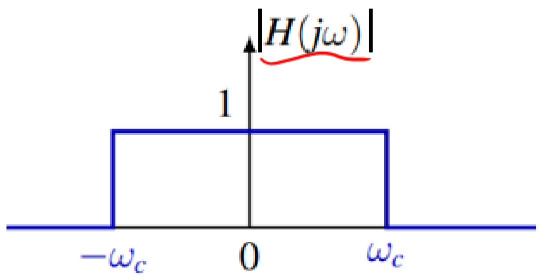


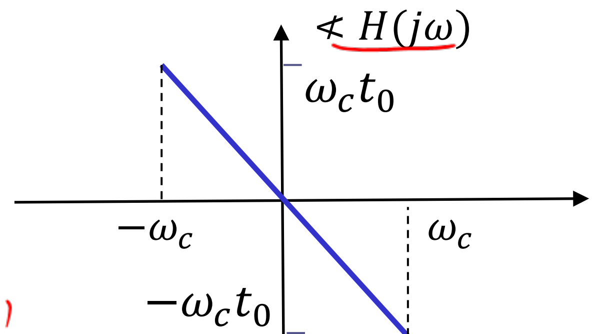
- Larger RC $\Longrightarrow$ passes smaller range of lower frequencies

Ideal Linear Phase LPF

- **Ideal lowpass filter** (unit magnitude, linear phase)

$$H(j\omega) = \begin{cases} e^{-j\omega t_0}, & |\omega| \le \omega_c \\ 0, & \text{otherwise} \end{cases}$$





$$\Rightarrow |H(j\omega)| = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c \end{cases}, \quad \measuredangle H(j\omega) = \begin{cases} -\omega t_0 & |\omega| \le \omega_c \\ 0, & \text{otherwise} \end{cases}$$

|H(jω)|

ωc

- $Y(j\omega) = e^{-j\omega t_0} X_{LP}(j\omega) \Rightarrow y(t) = x_{LP}(t - t_0)$
  - **Linear phase** ⇔ simply a **shift in time**, no distortion
  - Nonlinear phase ⇔ distortion in addition to time shift

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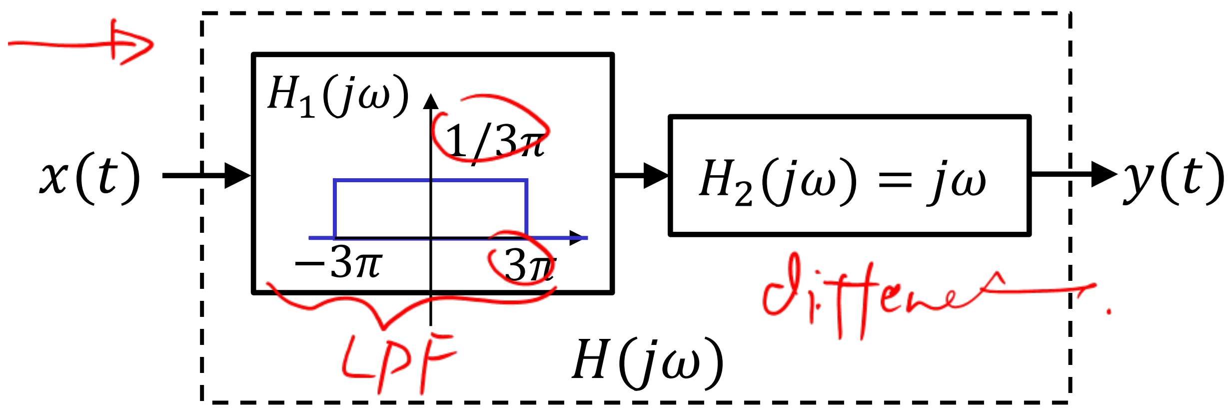
Example

$H_1(j\omega)\cdot j\omega$, $H_1(j\omega) = \begin{cases} \frac{1}{3\pi}, & |\omega| \le 3\pi \\ 0, & \text{otherwise} \end{cases}$

- **Frequency response** of an LTI system is

$$H(j\omega) = \begin{cases} \dfrac{j\omega}{3\pi}, & |\omega| \le 3\pi \\ 0, & \text{otherwise} \end{cases}$$

- Considered as a cascade of an LPF and a differentiator



Multiplication Property

modulation

- **Dual of convolution property:**

$$x(t)\cdot y(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}[X(j\omega) * Y(j\omega)]$$

multiplication in time ⇔ convolution in frequency

- **Proof:** let $Z(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$

$$\mathcal{F}^{-1}\{Z(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta\right]e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(j(\omega-\theta))e^{j\omega t}d\omega\right]d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)y(t)e^{j\theta t}d\theta = x(t)y(t)$$

Example

- Define a signal as

$$x(t) = \frac{\sin(t) \cdot \sin(t/2)}{\pi t^2} \triangleq \pi x_1(t) x_2(t)$$

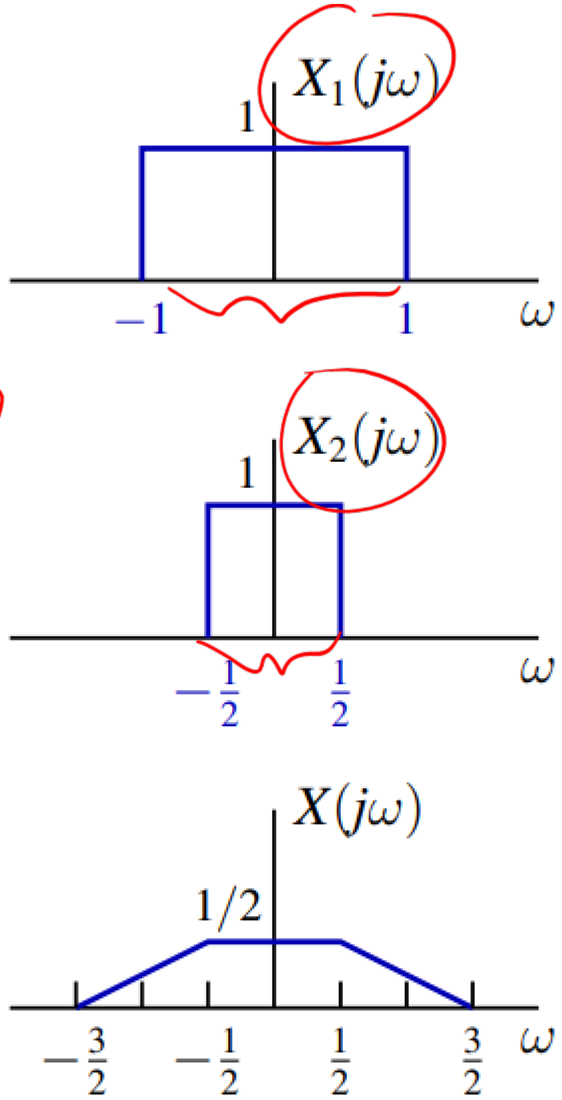
where,

$$x_1(t) = \frac{\sin(t)}{\pi t}$$

$$x_2(t) = \frac{\sin(t/2)}{\pi t}$$

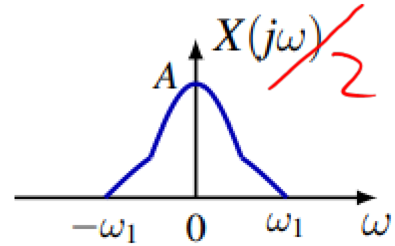
- Fourier transform

$$X(j\omega) = \frac{1}{2} X_1(j\omega) * X_2(j\omega)$$



Example: Modulation

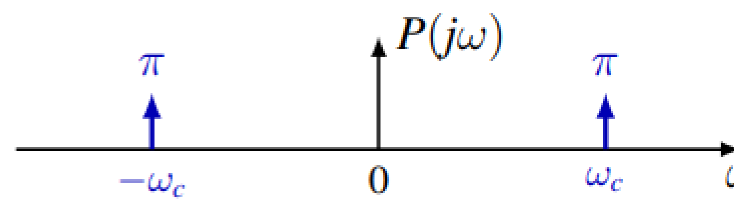
Baseband signal: $x(t)$



Carrier: $1 \longleftrightarrow 2\pi\delta(\omega)$

$$p(t) = \cos(\omega_c t) = \frac{1}{2}\left(e^{j\omega_c t} + e^{-j\omega_c t}\right)$$

$$P(j\omega) = \pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c)$$

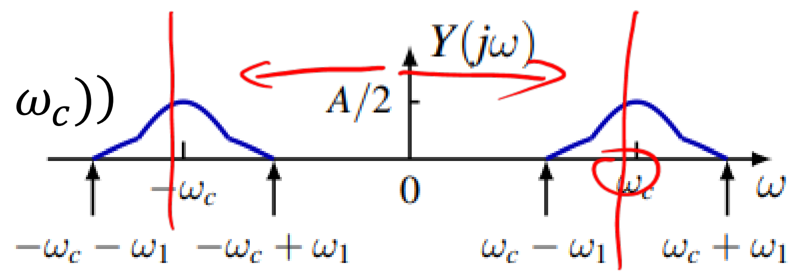


- **Modulated signal:**

$$y(t) = x(t)p(t)$$

$$Y(j\omega) = \frac{1}{2}X(j(\omega - \omega_c)) + \frac{1}{2}X(j(\omega + \omega_c))$$

$$\frac{1}{2\pi} X(j\omega) * P(j\omega)$$



Example: Demodulation

- **Modulated signal:**

  $y(t) = x(t)p(t)$

- **Carrier:**

  $p(t) = \cos(\omega_c t)$

- **Demodulated signal:**

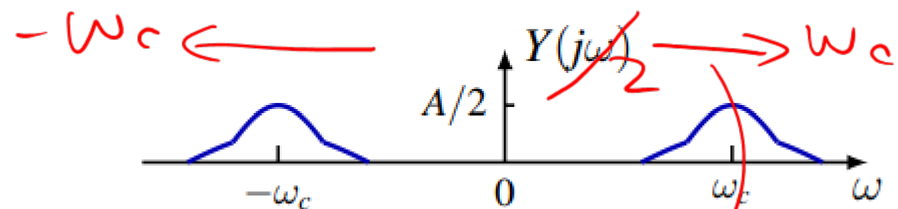
  $z(t) = y(t)p(t)$

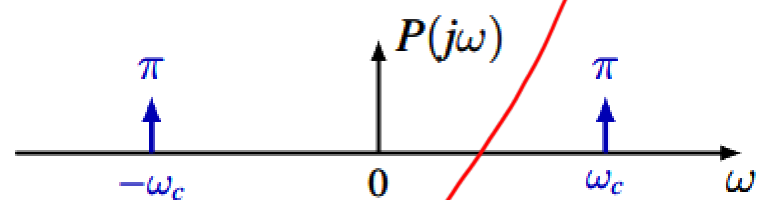
$$Z(j\omega) = \frac{1}{2}Y(j(\omega - \omega_c)) + \frac{1}{2}Y(j(\omega + \omega_c))$$

$$= \frac{1}{2}X(j\omega) + \frac{1}{4}[X(j(\omega - 2\omega_c)) + X(j(\omega + 2\omega_c))]$$

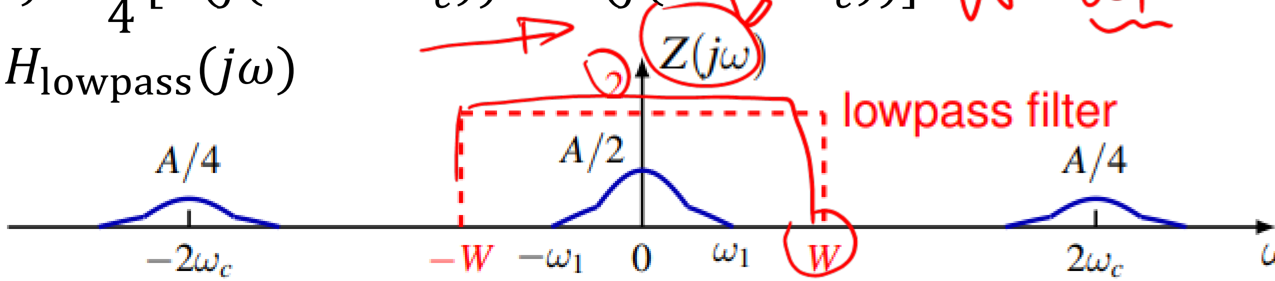
$R(j\omega) = Z(j\omega)H_{\text{lowpass}}(j\omega)$

$x(t) = r(t)$



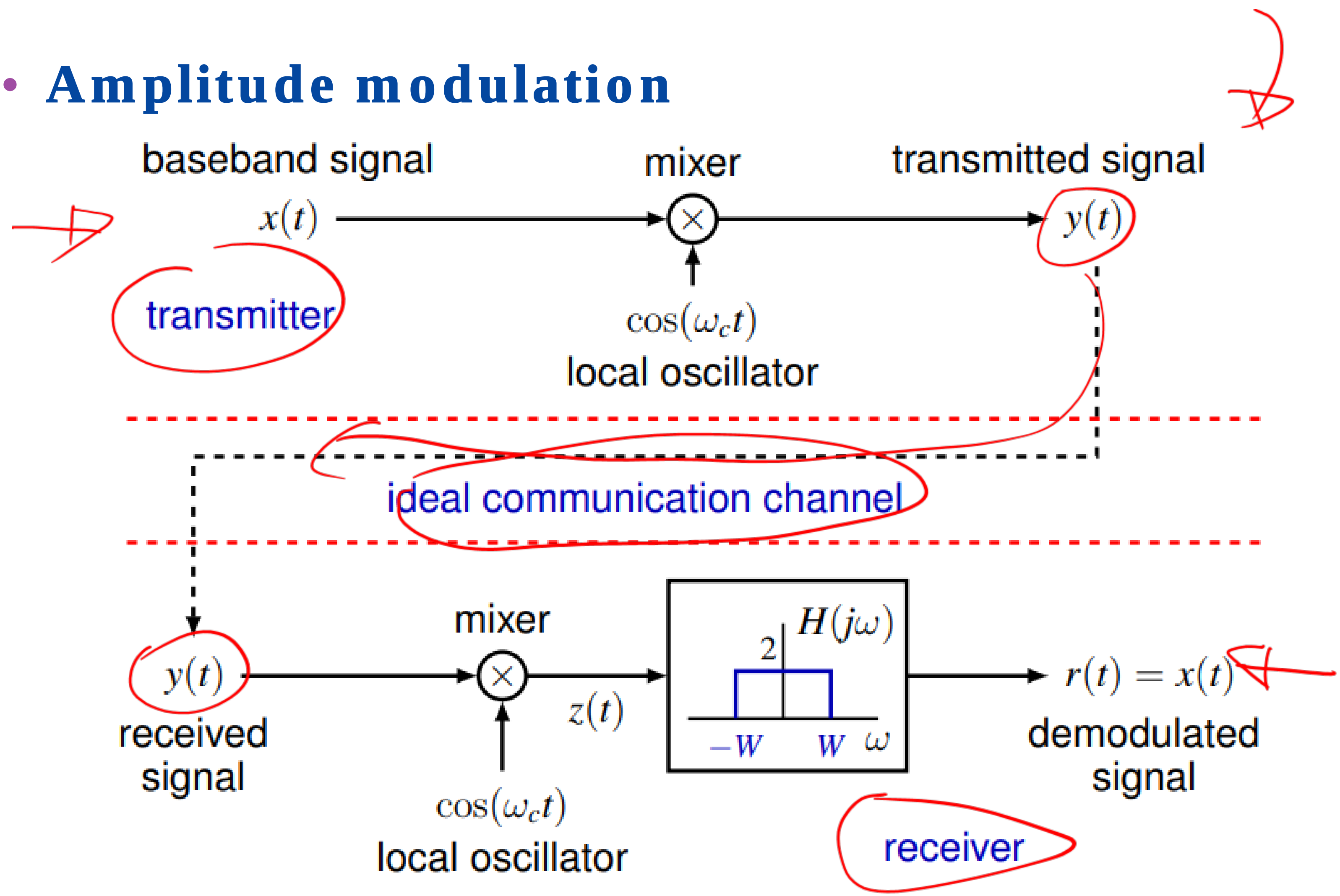


$W > \omega_1$



Ideal AM Communication System

- **Amplitude modulation**



LCCDEs

- Determine the **frequency response** of an LTI system described by

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- **Method 1:** using the **eigenfunction property**

let $\quad x(t) = e^{j\omega t} \rightarrow y(t) = H(j\omega)e^{j\omega t}$

substitution in to the differential equation yields

$$\sum_{k=0}^{N} a_k H(j\omega)(j\omega)^k e^{j\omega t} = \sum_{k=0}^{M} b_k (j\omega)^k e^{j\omega t}$$

$$H(j\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

LCCDEs

- **Method 2:** taking the **Fourier transform** of both sides

$$\mathcal{F}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathcal{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}$$

- By **linearity** and **differentiation in time property**

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

- $H(j\omega)$ is a **rational function** of $(j\omega)$, i.e., ratio of polynomials

Example

$v=-1, \quad v=-3$

$(v+1)H(v)\Big|_{\substack{v+1=0\\ v=-1}} = \dfrac{v+2}{v+3} = A_1 + \dfrac{A_2\cancel{(v+1)}}{v+3}$

$$y''(t) + 4y''(t) + 3y(t) = x'(t) + 2x(t)$$

- **Frequency response**

$$H(j\omega) = \frac{j\omega + 2}{(j\omega)^2 + 4(j\omega) + 3}$$

  - Using **partial fraction expansion**

$$H(j\omega) = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)} = \frac{A_1}{j\omega + 1} + \frac{A_2}{j\omega + 3}$$

  - Let $v = j\omega$

$$A_1 = (v+1)H(v)\Big|_{v=-1} = \frac{1}{2}, A_2 = (v+3)H(v)\Big|_{v=-3} = \frac{1}{2}$$

  - Taking inverse Fourier transform to find **impulse response**

$$h(t) = \left(\frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}\right)u(t)$$

Example

$$y''(t) + 4y''(t) + 3y(t) = x'(t) + 2x(t)$$

- Determine **zero-state response** to $x(t) = e^{-t}u(t)$ $\longleftrightarrow X(j\omega) = \frac{1}{j\omega + 1}$

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)} \cdot \frac{1}{j\omega + 1} = \frac{A_{1,1}}{j\omega + 1} + \frac{A_{1,2}}{(j\omega + 1)^2} - \frac{A_2}{j\omega + 3}$$

  - Using **partial fraction expansion**, let $v = j\omega$

$$A_{1,1} = \frac{1}{(2-1)!}\frac{d}{dv}\left[(v+1)^2 Y(v)\right]\Big|_{v=-1} = \frac{d}{dv}\left[\frac{v+2}{v+3}\right]\Big|_{v=-1} = \frac{1}{4}$$

$$A_{1,2} = (v+1)^2 Y(v)\Big|_{v=-1} = \frac{1}{2}, A_2 = (v+3)Y(v)\Big|_{v=-3} = -\frac{1}{4}$$

  - Taking inverse Fourier transform to find **output**

$$y(t) = \left(\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-3t}\right)u(t)$$

Q & A

Many Thanks